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ORIGINAL PAPER

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Piecewise constant response of underdamped oscillators through suppression of overshoots and undershoots in aerospace, civil, and mechanical systems

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Abstract When a single-degree-of-freedom underdamped system is subjected to a step function force in order that it eventually acquires a constant displacement, its response shows undershoots and overshoots. Since a step function force is difficult, if not impossible, to apply to many aerospace, civil, and mechanical engineering systems because of their inertias, this paper looks at simple forces that can be generated from a practical standpoint and are not instantaneously applied, that cause an underdamped oscillator to acquire a constant displacement *without* any overshoots and/or undershoots. These forces are ramped up (or down) over a short duration of time and held constant thereafter. A preliminary approach to the development of such force–time histories is presented by using a force given by a polynomial in time. Open-loop optimal control is next considered, and then closed-loop optimal control. The optimal control problem does not fall within the standard rubric of terminal control and a new approach for doing this is developed. These ideas are then woven into the development of a methodology that allows an undamped/underdamped single degree of freedom system to track a desired piecewise constant displacement time-history using forces that do not need to be instantaneously applied and that generate rippleless responses with no overshoots/undershoots. The methodology is also applicable to classically damped multi-degree-of-freedom systems with underdamped modes of vibration.

1 Introduction

This paper deals with the determination of practically usable forces that allow a linear underdamped single-degree-of-freedom (SDOF) mechanical system to follow a time history of piecewise constant displacements without any overshoots or undershoots. The desired response being piecewise constant over each subinterval of time, one is naturally drawn to the study of the response of such a system to a step input.

However, in most aerospace, civil, and mechanical engineering systems the control inputs being forces, the generation of forces that fit the description of a step input is often difficult to realize from a practical standpoint because of the system's inertia. Often, in practice, it is indeed very difficult, if not impossible, to generate abrupt, instantaneous forces, especially if they are required to be large.

The problem discussed in this paper is not restricted only to mechanical components modeled as single-degree-of-freedom (SDOF) systems, since a vibrating multi-degree-of-freedom (MDOF) system that is approximated well enough by a classically damped system—and numerous physical systems today are modeled in this way—would also have modes that would be caused to suffer in a similar manner.

There are other application areas too wherein a system is required to track a desired piecewise constant time-history without creating overshoots/undershoots, such as in the power industry for switching smoothly

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between different constant power level outputs during, say, the course of a day. Process industries where constant levels of chemicals need to be mixed/produced also encounter similar needs.

The paper is divided into two conceptual parts. The first part considers the elemental problem of tracking a desired constant displacement which the oscillator does not already have. Specifically, it considers the problem of finding the control force *over an interval of time* to steer the motion of an underdamped oscillator that starts with some initial velocity and displacement (often, both zero) so that it achieves a desired final constant amplitude of displacement, x_f. Depending on the force generation capabilities of the controller at hand, one might further prefer that this final desired constant displacement x_f be reached as soon as possible, and thereafter exactly maintained; or, that the constant displacement be reached in a time interval that is less than, or equal to, some specified value, and be exactly maintained thereafter. Furthermore, it is desirable that the response smoothly and monotonically gets to its desired constant final value, x_f, with no overshoots/undershoots. This is because a large overshoot in the oscillator's response above x_f, especially when it is lightly damped, can have significant ill-effects. It could over-stress the SDOF system leading to mechanical failure and/or the impairment of its function. Often, it may also affect the behavior of the overall system of which the SDOF system is a component, since the larger system may not be able to function properly until the response of the component in question settles down in a close enough vicinity of its desired final state.

There has been a considerable amount of work in the area of preventing/reducing overshoots and undershoots produced by step functions in systems modeled by a general set of linear differential equations. A good review of various PID controller designs for such systems subjected to a step function input may be found in Ref. [1], and the numerous papers this reference cites. Various techniques have been used for controllers that are tuned/optimized by different approaches, using state-space and/or pole-frequency methods. General conditions for no overshoots for linear systems subjected to step inputs have been found by Phillips and Seborg [2], Hara et al. [3], and Vidyasagar [4]. However, as pointed out earlier, step function forces are difficult to generate in mechanical systems and therefore this paper addresses the question of having an SDOF system reach a constant displacement value with no undershoots or overshoots through the use of forces that are not instantaneous but applied over a specified interval of time.

In this paper, the necessary and sufficient conditions are obtained for the force to result in a constant displacement at, and beyond, a pre-specified period of time T. These conditions specify restrictions on both the displacement and the force at time T. Forces that are smooth and that produce no overshoots are then developed using three approaches. The first is a polynomial force that satisfies the necessary and sufficient conditions. This is a preliminary step that obtains the force in closed form and illustrates the realizability of such a 'control' force. It is akin to the use of input shaping [5], though here this is done in the time domain and *exactly* engenders a constant response at, and beyond, the time T, with no overshoots/undershoots. Next, optimal control is utilized to obtain a desired force so that the oscillator starting from rest reaches a constant amplitude of response at, and beyond, time T while ensuring no overshoot beyond time T. Here the constraint on the magnitude of the control force at time T poses a difficulty since it places a constraint on the optimal control force over a set of measure zero. This places the problem beyond a straightforward terminal-state optimal control problem and points to the possible reason why this does not appear to have been done hereto. In this paper, a simple approach for handling this constrained terminal control problem in which the desired control force is required to have a specific value at the terminal time is presented. Both open-loop and feedback optimal control approaches are developed.

In the second part, in this paper, the understanding of this elemental problem is extended to the development of a control methodology to have an underdamped oscillator track a piecewise constant displacement time-history with no overshoots and undershoots using control forces that are not instantaneous but are applied over a prespecified duration of time.

The structure of this paper is as follows. Section 2 begins with the response of an undamped oscillator. It shows that it is possible to generate a simple force–time history such that the force is gradually ramped up (or down) over a period of time and then held constant, so that an undamped oscillator starting from rest can achieve a constant displacement with *no* overshoots or undershoots. It is shown that the duration of the linear (in time) ramp up (down) must equal the natural period of vibration of the oscillator (or a multiple of it). Section 3 provides the necessary and sufficient conditions for an underdamped oscillator starting from rest to have a constant displacement after a duration, T, of time. Starting with a preliminary approach by using a force that is polynomial in time, Sect. 4 develops optimal control approaches to generating practical force–time histories that allow an underdamped system to acquire a constant displacement after a prespecified duration of time with no overshoots and undershoots in its entire response. The problem of having constraints on the control force over sets of measure zero leads to standard optimal control formulations becoming inapplicable.

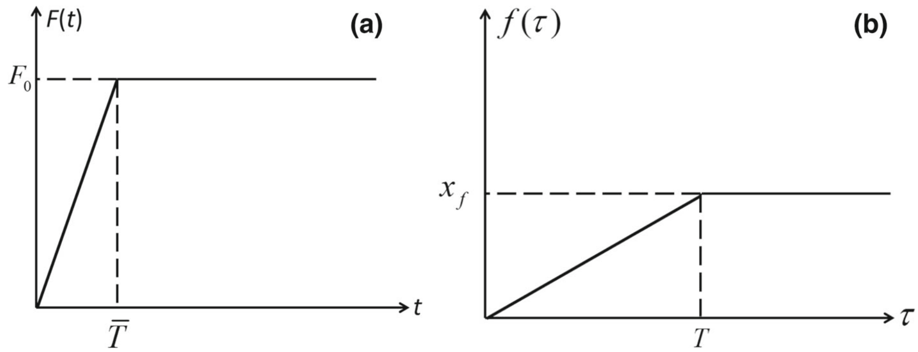

Fig. 1 **a** Force $F(t)$ ramps up linearly in time to its final value $F_0 = kx_{\mathrm{f}}$ and remains constant for $t \geq \bar{T}$. **b** The equivalent force $f(\tau)$ on the oscillator described by Eqs. (3) and (4), $T = \omega_n \bar{T}$

A novel approach is presented to circumvent this difficulty, and both open-loop and closed-loop optimal control techniques are thereby developed. Section 5 demonstrates the ease and efficacy with which the closed-loop optimal control obtained herein enables an underdamped oscillator to track a piecewise-constant displacement time-history with no overshoots or undershoots.

2 The undamped oscillator

To gain some initial insight, consider an undamped oscillator with mass m and stiffness k. We assume that at time $t = 0$, the displacement and the velocity of the oscillator are both zero so that $x(0) = \dot{x}(0) = 0$. Instead of applying the force F_0 abruptly (as with a step function) we consider applying the force more gradually as a (linear) ramp over a time duration $t = \bar{T}$, starting at zero and reaching its final value, F_0, at time $\bar{T}$, and thereafter remaining constant, as shown in Fig. 1a.

Thus, the oscillator described by the equation

$$m\ddot{x} + kx = F(t), \quad x(t = 0) = 0, \quad \dot{x}(t = 0) = 0 \tag{1}$$

is subjected to the forcing function

$$F(t) = \begin{cases} (F_0/\bar{T})\, t, & 0 \leq t \leq \bar{T} \\ F_0, & t \geq \bar{T} \end{cases} \tag{2}$$

instead of a step function force, which, as mentioned earlier, cannot be generated from a practical standpoint in most mechanical systems. Since we want the force in the time interval $[0, \bar{T}]$ to be most often simple to generate, we start with one that linearly changes with time. Hence, in the interval $t \in [0, \bar{T}]$ the force $F(t)$ is linear in time; it reaches its final value F_0 at time $\bar{T}$; thereafter, it keeps constant (see Fig. 1a). Dots denote derivatives with respect to time, t.

The intention is to steer the system by subjecting it to a force that yields a desired final (for $t > \bar{T}$) constant displacement, x_{f}. When the final displacement of the oscillator is constant for $t \geq \bar{T}$, $\dot{x}(t) \equiv 0$ so that $\ddot{x}(t) = 0$, and from Eq. (1) we obtain $x(t) = x_{\mathrm{f}} = F_0/k$. Hence, given a value of x_{f}, the magnitude of the force F_0 in Eq. (2) is simply $F_0 = kx_{\mathrm{f}}$. Knowing that the final force level must be F_0, the slope of the ramp in Fig. 1a must be adjusted so that at $t = \bar{T}$, $F(\bar{T}) = F_0$ as shown in Fig. 1a. The natural frequency of oscillation of the oscillator described in Eq. (1) is $\omega_n = \sqrt{k/m}$ and its period $T_n = 2\pi/\omega_n$.

In order to eliminate dependence on the natural frequency of oscillation, ω_n, we divide Eq. (1) by m and consider the dimensionless time $\tau = \omega_n t$. Equations (1) and (2) are thus written alternatively as

$$\frac{\mathrm{d}^2 x}{\mathrm{d}\tau^2} + x = \frac{F(\tau)}{k} := f(\tau), \quad x(\tau = 0) = 0,\ x'(\tau = 0) = 0, \tag{3}$$

with

$$f(\tau) = \begin{cases} \frac{F_0}{k}\frac{\tau}{T} = x_{\mathrm{f}}\frac{\tau}{T}, & 0 \leq \tau \leq T, \\ x_{\mathrm{f}}, & \tau \geq T, \end{cases} \tag{4}$$

where $T = \omega_n \bar{T}$ and the prime denotes differentiation with respect to τ. In what follows, we shall often use the system described by Eqs. (3) and (4) since it has the advantage that the normalized frequency of vibration of the system described by Eq. (3) is always unity and so its natural period $T_N = 2\pi$; the capital N subscript stands for the normalized period, with respect to the dimensionless time τ. See Fig. 1b. Alternatively stated, the use of a scaled time τ removes dependence on the natural frequency of vibration by normalizing ω_n to unity. Note that

$$T/T_N = \omega_n \bar{T}/T_N = \omega_n \bar{T}/2\pi = \bar{T}/(2\pi/\omega_n) = \bar{T}/T_n. \tag{5}$$

Our aim is to find a suitable value of the time T during which the force $f(\tau)$ gradually increases with time so that the response of this undamped system is steered in such a manner that it gets to the desired final displacement x_f with no overshoots/undershoots. Upon solving Eq. (3) using Eq. (4) for $0 \le \tau \le T$, it is easy to show that the displacement $x(\tau)$ and the velocity $x'(\tau)$ are given by

$$x(\tau) = \frac{x_\mathrm{f}}{T}[\tau - \sin(\tau)], \quad \text{and} \quad x'(\tau) := \frac{\mathrm{d}x}{\mathrm{d}\tau} = \frac{x_\mathrm{f}}{T}[1 - \cos(\tau)], \quad 0 \le \tau \le T \tag{6}$$

so that

$$x(T) = x_\mathrm{f}\left[1 - \frac{\sin(T)}{T}\right], \quad \text{and} \quad x'(T) = \frac{x_\mathrm{f}}{T}[1 - \cos(T)]. \tag{7}$$

Using these values of the displacement and velocity at time T, the response of system (3) for $\tau \ge T$ (during which the force $f(\tau)$ is held constant and equal to x_f, see Fig. 1b) can now be simply written down as

$$x(\tau) = x(T)\cos(\tau - T) + x'(T)\sin(\tau - T) + x_\mathrm{f}\{1 - \cos[(\tau - T)]\}, \quad \tau \ge T, \tag{8}$$

which upon using the expressions for $x(\tau = T)$ and $x'(\tau = T)$ from Eq. (7) becomes

$$x(\tau) = x_\mathrm{f} + \frac{x_\mathrm{f}}{T}\sin(\tau - T) - \frac{x_\mathrm{f}}{T}\underbrace{\{\sin(T)\cos(\tau - T) + \cos(T)\sin(\tau - T)\}}_{=\sin(\tau)}, \tag{9}$$

so that

$$x(\tau) = x_\mathrm{f} + \frac{x_\mathrm{f}}{T}[\sin(\tau - T) - \sin(\tau)], \text{ for } \tau \ge T. \tag{10}$$

From Eq. (10) we observe that when

$$T = r\,(2\pi) := rT_N, \quad r = 1, 2, \ldots, \tag{11}$$

where $T_N = 2\pi$ is the (normalized) period of the oscillator, then

$$x(\tau) = x_\mathrm{f} = F_0/k, \quad \text{for} \quad \tau \ge T\,! \tag{12}$$

What Eqs. (11) and (12) say is that, somewhat surprisingly, if the duration, T, in the ramp shown in Fig. 1b is any nonzero multiple of the normalized natural period $T_N (= 2\pi)$ of the undamped oscillator described by Eqs. (3) and (4), its response $x(\tau)$ is a *constant* equal to x_f for *any (all) time* τ that (equals or) exceeds this duration T shown in Fig. 1b! Of equal importance is the observation that the response $x(\tau)$ of the oscillator for $0 \le \tau \le T$ is given in Eq. (6), which is seen to be a monotone increasing function of τ, since $x'(\tau) \ge 0$ for all $0 \le \tau \le T$. In short, the response $x(\tau)$ therefore shows *no* oscillations in the interval $0 \le \tau \le T$!

In retrospect, we could have obtained this result in Eqs. (11) and (12) in a much simpler way. From Eq. (7) we see that when $T = rT_N = 2\pi r$, then $x(\tau = T) = x_\mathrm{f}$ and $x'(\tau = T) = 0$. Using Eq. (8) then trivially leads to the result given in Eq. (12). In the next section, we shall show that these conditions on x and x' will prevail even when the oscillator is damped.

Alternatively stated, Eqs. (11) and (12) inform us that when the force $F(t)$ (see Eqs. (1) and (2)) is ramped up starting from zero (see Fig. 1a) over a duration of time (note that $\bar{T} = T/\omega_n$)

$$\bar{T} = rT_N/\omega_n = r(2\pi/\omega_n) = rT_n, \quad r = 1, 2, \ldots \tag{13}$$

so that it reaches a value of $F_0 = x_\mathrm{f}k$ at time $\bar{T} = rT_n$ and thereafter remains constant, then the displacement of the oscillator $x(t)$ remains constant and equal to x_f for all $t \ge \bar{T}$. From the response $x(\tau)$ for $0 \le \tau \le T = rT_N$,

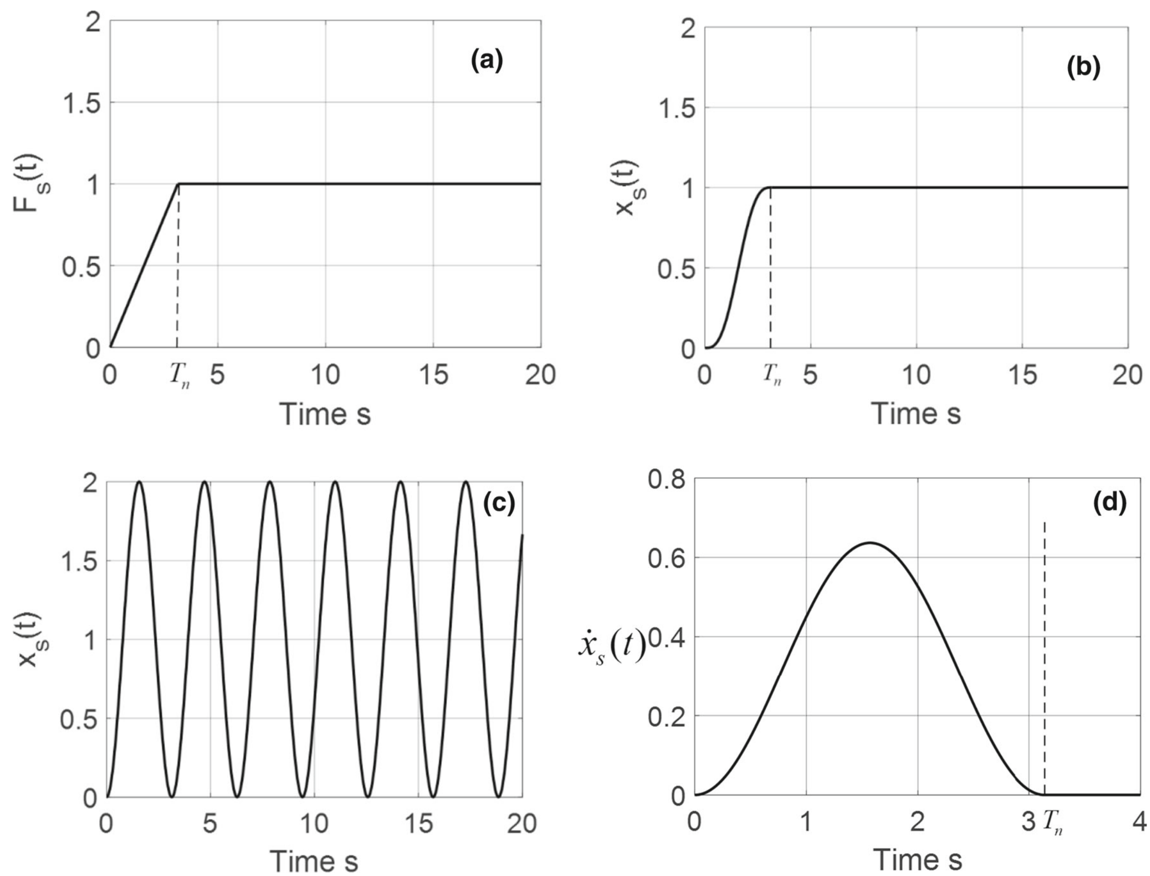

Fig. 2 $m = 2, k = 8, F_0 = 16$. **a** Scaled ramp function, **b** scaled response to ramp, **c** scaled response to step function, **d** scaled response, $\dot{x}_s(t)$ to ramp

one can directly infer the behavior of $x(t)$ for $0 \le t \le \bar{T} = rT_n$. In the time interval $[0, \bar{T}]$, $x(t)$ is a monotone increasing function with $\dot{x}(t) \ge 0$.

From a practical standpoint, since our aim most often is to steer the oscillator to reach a desired final constant displacement x_f as soon as possible, one would usually choose r in Eqs. (11) (or (13)) to be unity.

Of course, since the SDOF system (Eq. (1)) has no damping, its response $x(t)$ to the application of a step function force at $t = 0$ continues forever. One can think of the response as having an overshoot beyond its constant (static) displacement value that never decays because of the elephantine memory of the undamped oscillator. The response of the undamped system to such a step function (applied at $t = 0$) of magnitude F_0 is simply obtained as (assuming again $x(0) = \dot{x}(0) = 0$)

$$x(t) = \frac{F_0}{k}[1 - \cos(\omega_n t)], \tag{14}$$

whose *mean value* over a period (T_n) of the oscillator is the desired final displacement value $x_f = F_0/k$. Equation (14) also shows that the overshoot is twice the static displacement value, which is of course well known.

We now show a comparison between the response of an undamped oscillator (Eqs. (1) and (2)) to: (i) a step function of magnitude F_0 (applied at $t = 0$) and (ii) the ramp function $F(t)$ given in Eq. (2) and shown in Fig. 1, with $\bar{T} = T_n$, T_n being the natural period of the oscillator (we choose $r = 1$ in Eq. (13)). The numerical simulation is done using Matlab's platform for mass $m = 2$, stiffness $k = 8$, and a desired final constant displacement $x_f = 2$, all in consistent units. The frequency of vibration of this oscillator is 2 rad/s and its period T_n is π s. The force required to generate the desired final displacement is $F_0 = kx_f = 16$. The initial (at $t = 0$) displacement and velocity of the oscillator are both zero.

Figure 2a gives the scaled ramp function $F_s(t) = F(t)/F_0$ shown in Fig. 1 with $\bar{T} = T_n = \pi s$; Fig. 2b shows the scaled response $x_s(t) = x(t)/x_f$ of the oscillator to this ramp function. In these figures, and those to follow, the subscript 's' is used to indicate that the force $F(t)$ on the oscillator is scaled by F_0, and its displacement response $x(t)$ is scaled by x_f. For comparison, the scaled response of the oscillator to a step function force (applied at time $t = 0$) whose scaled magnitude is unity (i.e., $F_0 = 16$) is shown in Fig. 2c. Figure 2d shows that $\dot{x}_s(t) > 0$ in the interval $[0, T_n]$, except at its endpoints, where it is zero.

The response to the ramp function shows *no* overshoot or undershoot for $0 \le t \le T_n$, as expected, since $\dot{x}(t) > 0$ in the interior of the interval $[0, T_n]$, with $\dot{x}(0) = \dot{x}(T_n) = 0$. For values of $t \ge T_n$, the displacement response is a constant and has the desired final constant value $x_\mathrm{f} = 2$, as expected from Eqs. (11) and (12). As seen in Fig. 2b, the (displacement) response (i) starts at zero at time $t = 0$, with zero velocity, (ii) monotonically increases to reach the level x_f again with zero velocity, and (ii) then remains a constant, as shown in Fig. 2b. Not only is a step function force difficult to generate in mechanical systems, it causes the displacement of an undamped oscillator (with zero initial conditions) to undergo an endless sequence of over- and undershoots (Fig. 2c), that prevent the response of the oscillator from settling down to the final desired constant displacement value, x_f.

This demonstrates that the oscillations in the response shown in Fig. 2c of the undamped oscillator (that has an infinite 'memory') to a step function force can be totally suppressed by using a suitable ramp as the forcing function, and the desired final constant displacement value x_f can be obtained for $t \ge T_n$ without the slightest ripple (oscillation) in its response! To the best of the author's knowledge, though simple and elementary, this result appears to have gone unnoticed in the existing literature on the theory of vibrations (e.g., Refs. [6–11]).

The force for achieving this is surprisingly simple: instead of abruptly applying a constant force $F_0 = kx_\mathrm{f}$ to the oscillator (i.e., applying a step function force of magnitude F_0) and thereby causing it to 'shudder' with endless overshoots and undershoots in its response (see Fig. 2c), all that is needed is to start the force from zero, as one would in any practical situation, and simply ramp it up linearly in time over a duration equal to the natural period T_n of the oscillator so that at time $t = T_n$ the force reaches its full value F_0! Thereafter, for $t \ge T_n$, the force is simply kept constant at this value, F_0 (see Fig. 2a).

This simple analysis, however, has three important drawbacks that must be considered.

(1) First, the force $F(t)$ obtained above relies on being able to exactly set $\bar{T} = T_n$ for the ramp function [Eq. (2)] shown in Fig. 2a, and in practical situations this may be difficult to achieve with exactitude. It would therefore be useful to inquire how things might change if we are unable to implement, because of some practical limitations, this time duration $\bar{T}$ as exactly as required by our input force–time history shown in Fig. 2a. We know that our differential equation of motion is linear, and that its solution is a continuous function of the parameters describing the system. Hence, in a practical real-life situation, were we to have a small error in implementing the exact value of $\bar{T}(= T_n)$ after which the force input $F(t)$ remains constant, the oscillator's response would be appropriately altered in a small and continuous fashion.

Figure 3 shows a simulation for the system considered in Fig. 2 with the force input $F(t)$ shown in Fig. 2, except that the value of $\bar{T}$ is not T_n as required [see Eq. (2)], but $\bar{T} = (1 \pm 0.05)T_n$. That is, we consider an error of $\pm 5\%$ in the practical implementation of the desired value of $\bar{T} = T_n$ when generating our applied force. The parameters describing the oscillator (m and k) are the same as those used to obtain Fig. 2. The response scaled by x_f, namely, $x_s(t) = x(t)/x_\mathrm{f}$ is shown. The dashed curve shows, for comparison, the response to a step function force of magnitude $F_0 = 16$ applied at $t = 0$.

It is easy to show that when $\bar{T} = (1 + \alpha)T_n$, where α is the relative error in implementing the proper value of $\bar{T}$, the expression corresponding to Eq. (10) becomes

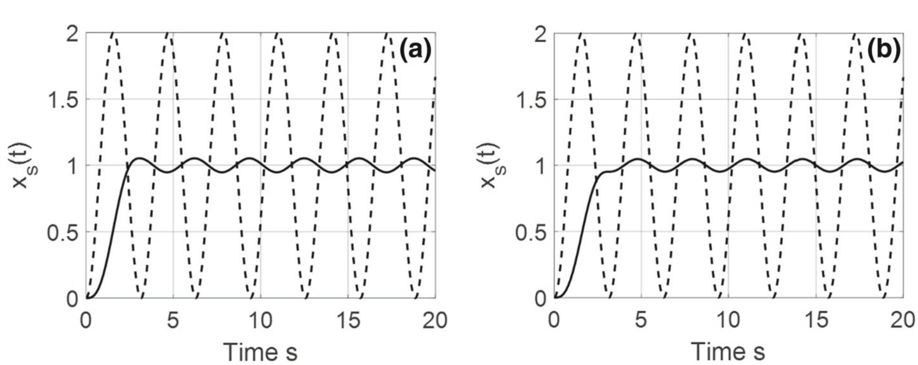

Fig. 3 Response to the ramp function with **a** $\bar{T} = 0.95T_n$, **b** $\bar{T} = 1.05T_n$

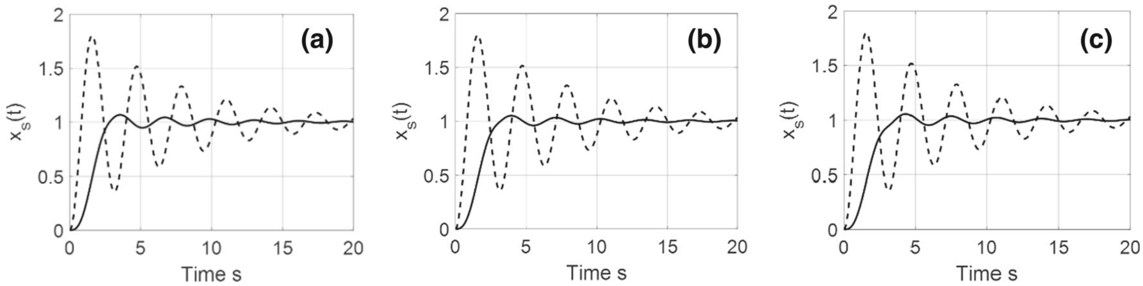

Fig. 4 Scaled response, $m = 2$, $k = 8$, $\zeta = 7\%$. **a** $\bar{T} = 0.95T_n$, **b** $\bar{T} = T_n$, **c** $\bar{T} = 1.05T_n$

$$x(t) = \frac{F_0}{k}\left[1 - \frac{\sin(\pi\alpha)}{\pi(1+\alpha)}\cos(\omega_n t - \pi\alpha)\right], \quad \text{for } t \geq \bar{T}. \tag{15}$$

Equation (15) shows that a ripple with frequency ω_n, phase angle $\pi\alpha$, and (scaled) amplitude $\frac{\sin(\pi\alpha)}{\pi(1+\alpha)} \approx \alpha - \alpha^2 + O(\alpha^3)$ is introduced in the response. As seen from Fig. 3, the (scaled) amplitude of this ripple in the response for $\alpha = 0.05$ is small and has a value of 0.0474.

(2) The second drawback of our simple analysis, which is equally important if not more so, is that undamped systems are idealizations, and real-life systems always have some amount of damping. Using the continuity argument stated before, if we were to have a damped system in which the percentage of critical damping is "small"—instead of being zero as in the undamped systems considered hereto—its response to our ramp function (with $\bar{T} = T_n$) would change from that shown in Fig. 2b in a small and continuous manner. Figure 4b shows the behavior of the underdamped system described by

$$m\ddot{x} + c\dot{x} + kx = F(t) = \begin{cases} (F_0/\bar{T})\,t, & 0 \leq t \leq \bar{T}, \\ F_0, & t \geq \bar{T}, \end{cases} \quad x(0) = \dot{x}(0) = 0, \tag{16}$$

where the critical damping $\zeta = c/(2m\omega_n) = 7\%$, with values, as before, of $m = 2$, $k = 8$, and $F_0 = 16$ (in consistent units), so that the desired final displacement is $x_f = F_0/k = 2$ (in consistent units). The scaled response $x_s(t) = x(t)/x_f$ is shown.

Again, in a practical implementation it might be difficult to set $\bar{T}$ exactly equal to T_n. Figure 4 shows simulations of the response (solid lines) of the damped system ($m = 2$, $k = 8$) to the ramp function (see Eq. (16)) in which $\bar{T} = 0.95T_n$, T_n, and $1.05T_n$. For comparison, in each figure, the dashed line (which is the same in all 3 figures) shows the damped ($\zeta = 7\%$) response to a step function of magnitude $F_0 = 16$ (applied at $t = 0$).

These figures show that the use of a ramp function can constitute a simple and practical way of steering a lightly damped SDOF oscillator to a desired final constant displacement value, assuming that its natural period is moderately small. The step function response of the damped oscillator shows a much larger overshoot than those for the ramp functions. Also, the step function response takes considerably greater time to get into the vicinity of the final desired constant displacement, $x_f = 2$.

Our reliance on the continuity of solutions of the differential equation (16) on the parameter ζ would work well when the value of $\zeta \ll 1$, which is a circumstance that often occurs in aerospace, mechanical, and civil engineering systems. While quantitative estimates of the differences between the response in Fig. 2b and those in Fig. 4 can be obtained by using perturbation methods, such results would still be applicable only when $\zeta \ll 1$ and $\left|\frac{\bar{T}}{T_n} - 1\right| \ll 1$. And so, we would need a more precise method that would be applicable for underdamped systems where $0 \leq \zeta < 1$.

(3) The third shortcoming of our simple analysis is that we have chosen the duration of the ramp forcing function $\bar{T} = T_n$ (or alternatively, $T = T_N = 2\pi$). One is usually interested in acquiring the desired final constant displacement response, x_f, quickly. What if the value of $\bar{T}$ is unacceptably large? What if the desired constant displacement x_f is required to be reached, for practical reasons, in a shorter span of time than T_n?

In the next section, we address these drawbacks. Our central aim is to device a simple force $F(t)$—not necessarily a linear ramp function as was done in Eq. (16)—that guides an underdamped oscillator to a desired final constant displacement state $x(t) = x_f$ without any overshoots or undershoots, given a prescribed value of $0 < \bar{T} \leq T_n$.

3 The underdamped oscillator

We consider an oscillator that starts with zero initial conditions with mass m, stiffness k, and percentage of critical damping ζ. From the previous section, two zones of response are now conceptually recognized. The first zone of response is conceptually typified by the interval $[0,\ \bar{T}]$ (see Fig. 1(a)) during which the force $F(t)$ changes from zero—though, in general, not linearly in time now—, thereby causing the displacement to also change from zero and reach the desired final displacement x_{f} at $t = \bar{T}$. The second zone is typified by the region $t \geq \bar{T}$ (Fig. 1(a)) during which the response remains at its desired final (constant) displacement value x_{f}. We begin by focusing first on the second zone of response.

Our aim is to let the oscillator, whose displacement and velocity are as yet unknown at time $t = \bar{T}$, have a given desired constant displacement x_{f} for $t \geq \bar{T}$, for some suitably prescribed time $\bar{T}$, when subjected to a suitable external force $F(t)$. This can be written as

$$m\ddot{x} + c\dot{x} + kx = F(t), \quad t \geq \bar{T} > 0 \tag{17}$$

with

$$x(\bar{T}) = a,\ \dot{x}(\bar{T}) = b. \tag{18}$$

The parameter c equals $2m\omega_n\zeta$ in Eq. (17). The damped period of vibration of the oscillator is denoted by $T_d = T_n/\sqrt{1-\zeta^2}$. We will be mainly interested in investigating $0 < \bar{T} \leq T_n$ (and $\bar{T} = T_d$).

Equations (17) and (18) are rewritten, as before, so that the normalized natural frequency of vibration of the oscillator is unity by using the dimensionless time $\tau = \omega_n t$. This yields the alternative relation

$$x'' + 2\zeta x' + x = f(\tau), \quad \tau \geq T > 0, \quad x(\tau = T) = a,\ x'(\tau = T) = b/\omega_n, \tag{19}$$

where $T = \omega_n \bar{T}$, the primes now denote differentiation with respect to τ, $0 \leq \zeta < 1$ is the percentage of critical damping of the underdamped system, and x_{f} is the desired final displacement. The normalized natural period of vibration of the oscillator in Eq. (19) is $T_N = 2\pi$, and its normalized damped period of vibration is $T_D = 2\pi/\sqrt{1-\zeta^2}$. We will presently find the constants a and b so that the oscillator described by Eq. (19) has a desired final constant displacement x_{f} for $\tau \geq T > 0$. Note that the corresponding constant external force applied to the SDOF oscillator in Eq. (17) is given by $F(t) = kf(\tau)$.

Since for all $\tau \geq T$, we want $x(\tau)$ to equal x_{f}, which is a constant, we must have $x'(\tau) = x''(\tau) = 0$ for $\tau \geq T$. Setting these derivatives to zero on the left hand side of Eq. (19) yields the relation $f(\tau) = x(\tau)$, so that $f(\tau) = x_{\mathrm{f}}$ for $\tau \geq T$ (or $F(t) = k\,x_{\mathrm{f}} := F_0$ in Eq. (17)).

The solution to Eq. (19) is then straightforward and is given by

$$x(\tau) - x_{\mathrm{f}} = [a - x_{\mathrm{f}}]\exp(-\zeta\tilde{\tau})\cos(S\tilde{\tau}) + \left[w(T) - \frac{x_{\mathrm{f}}\zeta}{S}\right]\exp(-\zeta\tilde{\tau})\sin(S\tilde{\tau}), \quad \tau \geq T, \tag{20}$$

where

$$\tilde{\tau} = \tau - T, \quad S = \sqrt{1-\zeta^2}, \quad \text{and} \quad w(T) = \frac{(b/\omega_n) + a\zeta}{S}. \tag{21}$$

Since the functions $\exp(-\zeta\tilde{\tau})\cos(S\tilde{\tau})$, and $\exp(-\zeta\tilde{\tau})\sin(S\tilde{\tau})$ are linearly independent for $\tau \in [T, \infty)$, and since we want the oscillator's response $x(\tau)$ to equal the desired final constant displacement response $x_{\mathrm{f}} = F_0/k$ for (all) $\tau \geq T$, we require from Eq. (20) that

$$a = x_{\mathrm{f}} \quad \text{and} \quad w(T) = \frac{x_{\mathrm{f}}\zeta}{S}. \tag{22}$$

Using Eq. (21) for $w(T)$ and the first relation in Eq. (22), we find from the second relation in Eq. (22) that

$$\frac{(b/\omega_n) + x_{\mathrm{f}}\zeta}{S} = \frac{x_{\mathrm{f}}\zeta}{S}\,[= w(T)],$$

so that $b = 0$, and therefore we require (see Eq. (19))

$$x(T) = x_{\mathrm{f}} \quad \text{and} \quad x'(T) = 0. \tag{23}$$

Hence, the SDOF system described by the left-hand side of Eq. (19) will have a final desired constant response $x(\tau) = x_{\mathrm{f}}$ for $\tau \geq T$ if

$$(1)\ \ f(\tau = T) = x_{\mathrm{f}} \quad \text{and,}\ \ (2)\ \ x(\tau = T) = x_{\mathrm{f}} \quad \text{and} \quad x'(\tau = T) = 0, \tag{24}$$

or alternatively (see Eqs. (17) and (18)),

$$(2)\ \ F(t = \bar{T}) = k\, x_{\mathrm{f}} \quad \text{and} \quad (2)\ \ x(t = \bar{T}) = x_{\mathrm{f}} \quad \text{and} \quad \dot{x}(t = \bar{T}) = 0. \tag{25}$$

Also, when the set of conditions in Eq. (24) is satisfied, Eq. (20) shows that $x(\tau) = x_{\mathrm{f}}$ for $\tau \geq T$. Hence, Eqs. (24) and (25) provide a set of necessary and sufficient conditions for $x(\tau) = x_{\mathrm{f}}$ for $\tau \geq T$.

We have thus described the conditions needed for the response of the oscillator to have a constant desired displacement for τ(or t) $\geq T$(or $\bar{T}$), that is, in our second zone of response.

We now consider the first zone of response where $\tau \in [0, T]$. Since no instantaneous increase in force is desired, because such an increase is difficult to generate in mechanical systems, we want that the force $f(\tau = 0) = 0$ (or alternatively, that $F(t = 0) = 0$). This is then a requirement that needs to be added to those stated in Eq. (24) (or, Eq. (25)).

Thus, in the first zone of response, i.e., in the interval $\tau \in [0, T]$, all we need to do is to steer the oscillator described by (the left hand side of) Eq. (19) from its initial state $x(0) = 0,\ x'(0) = 0$ to the state at time T given by $x(T) = x_{\mathrm{f}} = \frac{F_0}{k}$, and $x'(T) = 0$ by applying a suitable force $f(\tau)$ over the interval $[0, T]$. This steering force $f(\tau)$ has two characteristics (a) $f(0) = 0$, and (b) $f(T) = x_{\mathrm{f}}$. Upon reaching a value of x_{f} at time $\tau = T$, the force $f(\tau)$ thereafter remains constant at this value.

In summary, the idea then is to ramp up the force $f(\tau)$ in the interval $[0, T]$ so that $f(\tau = 0) = 0$ and $f(\tau = T) = x_{\mathrm{f}}$, while ensuring that $x(T) = x_{\mathrm{f}}$ and $x'(T) = 0$. This can be done in numerous ways.

4 Forces that produce no overshoots/undershoots in the response of underdamped oscillators

In this section, we consider different approaches to obtaining the requisite force $f(\tau)$ that ensures that the response of the oscillator reaches the desired final constant value x_{f} monotonically, with no overshoots or undershoots, given the interval of time $0 < T \leq T_N$ over which the oscillator is required to reach this final value. We begin with the simplest, preliminary approach that requires only an understanding of the elementary theory of vibrations by using a polynomial forcing function, then move to open-loop optimal control methods, and further on to robust closed-loop optimal control approaches that exactly engender responses with *no* ripples in them.

4.1 Polynomial forcing function—a preliminary approach

A very simple form of the force $f(\tau)$, $0 \leq \tau \leq T$, that is given by a polynomial in time is considered in this subsection. We thus have the system

$$x'' + 2\zeta x' + x = f(\tau), \quad x(0) = x'(0) = 0, \tag{26}$$

where

$$f(\tau) = \begin{cases} a_0 + b_0\tau + c_0\tau^2 + d_0\tau^3, & 0 \leq \tau \leq T, \\ x_{\mathrm{f}}, & \tau \geq T > 0. \end{cases} \tag{27}$$

As per Eq. (24), our aim is to steer the oscillator from its initial state $\{x(0) = 0, x'(0) = 0\}$ to its final state given by $\{x(T) = x_{\mathrm{f}}, x'(T) = 0\}$ in the interval $\tau \in [0, T]$. To ensure this we need to find the (constant) coefficients $a_0,\ b_0,\ c_0$ and d_0 appropriately.

In addition, we want $f(0) = 0$ and $f(\tau = T) = x_{\mathrm{f}}$. Using Eq. (27), the first of these two conditions requires that $a_0 = 0$, and the second requires that

$$b_0 = \frac{x_{\mathrm{f}}}{T} - c_0 T - d_0 T^2. \tag{28}$$

Thus, we need to find just the two independent constants, c_0 and d_0, in Eq. (27).

The two constants c_0 and d_0 in Eq. (27) can be found using elementary methods so that $x(\tau)$ and $x'(\tau)$ satisfy the conditions at times $\tau = 0$ and $\tau = T$.

Using Eq. (28), the response of the oscillator for $0 \le \tau \le T$ to the force described in Eq. (26) is thus described by the equation

$$x'' + 2\zeta x' + x = \left\{ \frac{x_{\mathrm{f}}}{T} - c_0 T - d_0 T^2 \right\} \tau + c_0 \tau^2 + d_0 \tau^3, \tag{29}$$

whose solution is easily obtained as ($S = \sqrt{1 - \zeta^2}$)

$$x(\tau) = A + B\tau + C\tau^2 + D\tau^3 + \exp(-\zeta\tau)[H_1 \cos(S\tau) + H_2 \sin(S\tau)], \tag{30}$$

where the constants

$$A = 2\left(T\zeta + 4\zeta^2 - 1\right) c_0 + 2\left(T^2\zeta - 24\zeta^3 + 12\zeta\right) d_0 - \frac{2\zeta x_{\mathrm{f}}}{T}, \tag{31}$$

$$B = -\left(T + 4\zeta\right) c_0 - \left(T^2 - 24\zeta^2 + 6\right) d_0 + \frac{x_{\mathrm{f}}}{T}, \quad C = c_0 - 6\zeta d_0, \quad \text{and} \quad D = d_0. \tag{32}$$

To enforce the initial conditions $x(0) = 0,\ x'(0) = 0$, we further require that

$$H_1 = -A, \quad \text{and} \quad H_2 = -\frac{(B + \zeta A)}{S}. \tag{33}$$

Using Eqs. (31)–(33) in Eq. (30) the explicit solution $x(\tau; \zeta, c_0, d_0, x_{\mathrm{f}})$ for $0 \le \tau \le T$ is obtained.

We next enforce the conditions at $\tau = T$, namely $x(T) = x_{\mathrm{f}}$ and $x'(T) = 0$. These two conditions give two simultaneous equations for the constants $c_0(\zeta, T, x_{\mathrm{f}})$ and $d_0(\zeta, T, x_{\mathrm{f}})$,

$$\underbrace{\begin{bmatrix} p_1 & p_2 \\ q_1 & q_2 \end{bmatrix}}_{A_0} \begin{bmatrix} c_0 \\ d_0 \end{bmatrix} = x_{\mathrm{f}} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}. \tag{34}$$

The constants $\{p_i, q_i, r_i\}$, $i = 1, 2$, which are functions solely of ζ and T, in Eq. (34) are explicitly given in Appendix 1.

Upon solving these simultaneous equations for c_0 and d_0 and using Eq. (28) to obtain b_0, the three coefficients $b_0,\ c_0,$ and d_0 (recall, $a_0 = 0$) that describe the force $f(\tau)$ given in Eq. (27) are determined. We note that $f(\tau = 0) = 0$ and $f(\tau = T) = x_{\mathrm{f}}$. And this force also ensures that the oscillator's displacement, $x(\tau = T) = x_{\mathrm{f}}$, and its velocity, $x'(\tau = T) = 0$. With these conditions on the state at $\tau = T$ (Eq. (24)), the constant force $f(\tau) = x_{\mathrm{f}}$, $\tau \ge T$, will keep the displacement of the system constant at a value of x_{f}.

From Eqs. (28) and (34), it is seen that the coefficients $b_0,\ c_0,$ and d_0, are each proportional to x_{f}. One can thus determine the coefficients $b_0,\ c_0,$ and d_0 using Eqs. (34) and (28) for $x_{\mathrm{f}} = 1$; to obtain the coefficients for any desired value to x_{f} in any specific situation each of these coefficients can simply be multiplied by the desired x_{f}. Also, for a given value of x_{f} these coefficients are *only* functions of ζ and T.

The solution of Eq. (34) requires that the determinant of the matrix A_0 be nonzero. This determinant, which is given explicitly in Appendix 1, depends only on the parameters T and ζ. Its Taylor series expansion at $T = 0$ is also provided in Appendix 1, and shows that its leading term is $O(T^8)$. The determinant is plotted for $T/T_N = T/2\pi \in [0.05,\ 1]$ and shown in Fig. 5a. Ten equally spaced points in this interval are taken, and for each such value of this ratio the determinant is found for values of $\zeta \in [0, 0.99]$. Each of these ten values of T/T_N generates a curve on the plot. As seen the determinant is nonzero; the determinant is bounded between the two solid black lines for $T/T_N = 0.05$ and $T/T_N = 1$. For $T/T_N = 0.01$, on the log10 scale shown in Fig. 5a the determinant ≈ -12.48. Figure 5b shows the logarithm of the determinant of A_0 for $T = T_D := T_N/\sqrt{1 - \zeta^2}$, which is the damped normalized period of the oscillator described by Eq. (26).

Once the triplet $\{b_0, c_0, d_0\}$ is obtained as above by solving Eq. (34) for the system described by Eqs. (26) and (27), results for our original (non-normalized) oscillator with mass m and stiffness k can be simply obtained, by noting that $\tau = \omega_n t$. Thus, given the description of an oscillator, i.e., the values of the triplet $\{m, k, 0 \le \zeta < 1\}$, and the desired final constant displacement x_{f}, one can explicitly and directly determine the force $F(t)$ required so that this system achieves the required desired displacement x_{f}. The equation describing

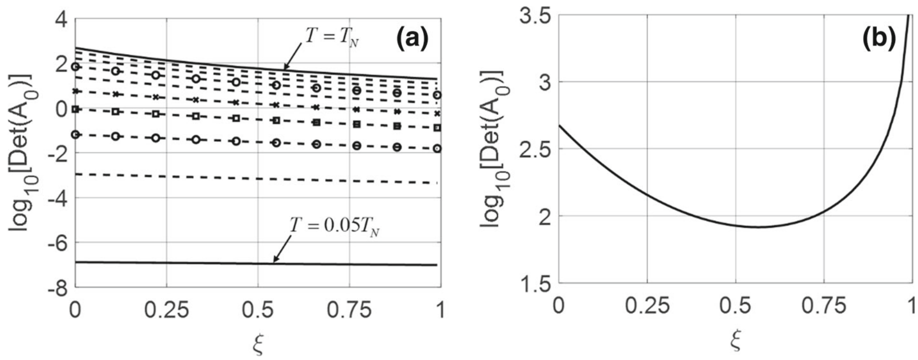

Fig. 5 **a** $\mathrm{Det}(A_0)$ versus ζ for $T/T_N \in [0.05, 1]$, **b** $\mathrm{Det}(A_0)$ versus ζ for $T = T_D := T_N/\sqrt{1-\zeta^2}$

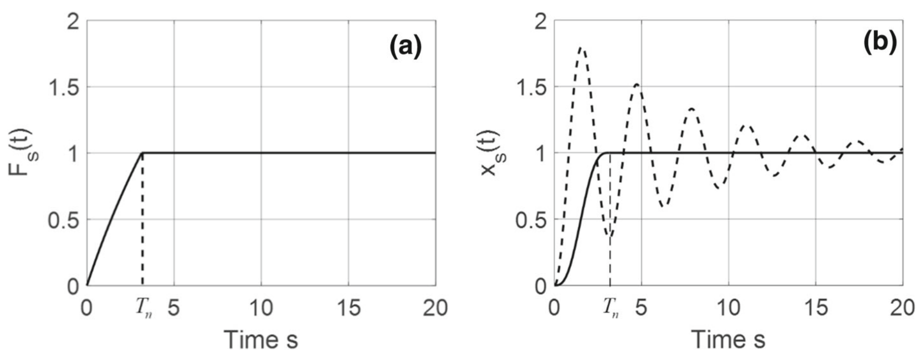

Fig. 6 $m = 2, k = 8, \zeta = 7\%$. **a** $F_s(t) = (B_0t + C_0t^2 + D_0t^3)/F_0$ in $t \in [0, T_n]$, **b** response to $F(t)$

its displacement $x(t)$ is directly obtained from its counterpart, Eqs. (26)–(27), and is given by the equation $(\bar{T} = T/\omega_n)$

$$m\ddot{x} + c\dot{x} + kx = F(t) := \begin{cases} B_0t + C_0t^2 + D_0t^3, & 0 \le t \le \bar{T}, \\ F_0 = k\,x_{\mathrm{f}} & t \ge \bar{T} > 0, \end{cases} \quad x(0) = \dot{x}(0) = 0 \tag{35}$$

It will yield the final desired constant displacement $x(t) = x_{\mathrm{f}}$ for $t \ge \bar{T} > 0$. The constants in Eq. (35) are

$$B_0 = k\omega_n b_0,\ C_0 = k\omega_n^2 c_0, \quad \text{and} \quad D_0 = k\omega_n^3 d_0 \tag{36}$$

with c_0, and d_0 obtained by solving Eq. (34), and b_0 obtained by using Eq. (28). With the constants $B_0, C_0,$ and D_0 so obtained, we are also assured that $x(\bar{T}) = x_{\mathrm{f}}$, and $\dot{x}(\bar{T}) = 0$.

It should be noted that as yet no choice has been made about the value of the parameter T in Eq. (27) or the corresponding parameter $\bar{T}$ in Eq. (35).

As an example, we again take an underdamped SDOF system considered before with $m = 2$, $k = 8$, and $\zeta = 0.07$ $(\omega_n = 2)$. The oscillator is assumed to start with zero displacement and velocity at $t = 0$. We assume that the desired final constant displacement required of the oscillator is $x_{\mathrm{f}} = 2$. Therefore $F_0 = kx_{\mathrm{f}}$. Taking a hint from the undamped oscillator considered in the previous section, we take the duration of time $\bar{T}$ (see Eq. (35)) over which the non-constant ramp force steers the oscillator to this final constant value of F_0 as $\bar{T} = 2\pi/\omega_n := T_n$, where T_n is the natural period of vibration of the oscillator (see Fig. 6a). The expression for the force $F(t)$ given in Eq. (35) now becomes

$$F(t) = \begin{cases} B_0t + C_0t^2 + D_0t^3, & 0 \le t \le T_n, \\ F_0, & t \ge T_n. \end{cases} \tag{37}$$

As noted, the constants $B_0, C_0,$ and D_0 are found using Eq. (36). The constants c_0 and d_0 are obtained by solving the simultaneous equations (34), with $T = 2\pi$; knowing these two constants, b_0 is determined as noted in Eq. (28). We then obtain $B_0 = 6.3301$, $C_0 = -0.4713$, and $D_0 = 0.0247$. The resulting (scaled) force

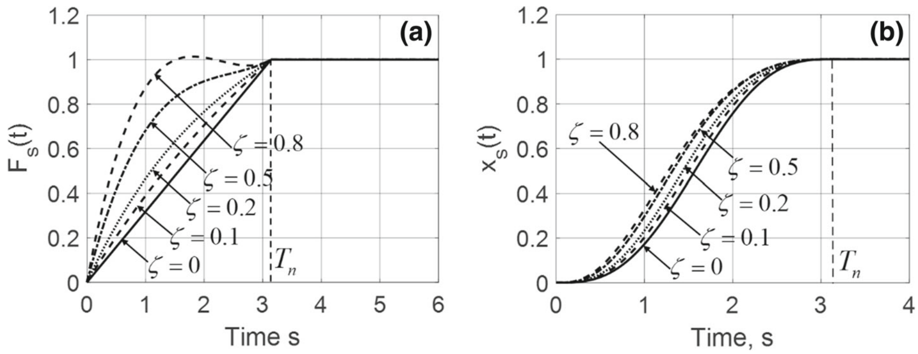

Fig. 7 $m = 2, k = 8$. **a** $F_s(t) = B_0 t + C_0 t^2 + D_0 t^3 / F_0$ in $t \in [0, T_n]$, **b** response to force $F(t)$

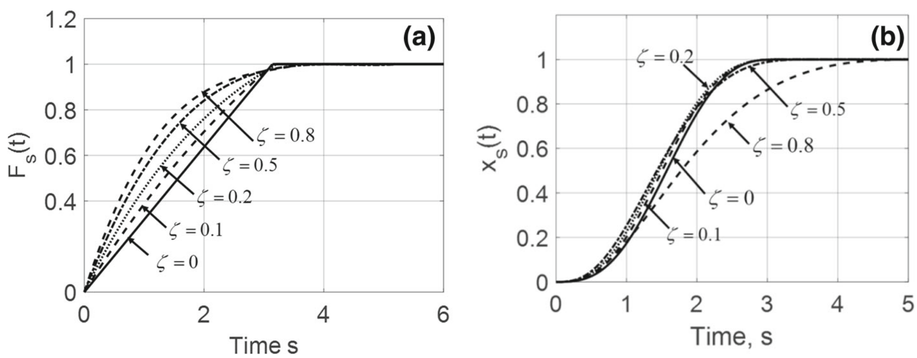

Fig. 8 $m = 2, k = 8$. **a** $F_s(t) = B_0 t + C_0 t^2 + D_0 t^3$ in $t \in [0, T_d]$, **b** response to force $F(t)$

$F_s(t) = F(t)/F_0$ is shown plotted in Fig. 6a. Note that $F_s(T_n) = 1$. The damping being 'small', the force $F(t)$ in the time interval $[0, T_n]$ resembles a straight line as in Fig. 1, though it is not quite straight. While the coefficient, B_0, of the linear term in Eq. (37) is the largest, we find that $|B_0/C_0| \approx 13.4$ and $B_0/D_0 \approx 256.3$.

The oscillator's response (solid line), which has been scaled by the desired steady state (static) response $x_\mathrm{f} = F_0/k$ so that $x_s(t) = x(t)/x_\mathrm{f}$, is shown in Fig. 6b. For comparison, the scaled response of the same oscillator subjected to a step function force F_0 (applied for $t \geq 0$) is shown by the dashed line in Fig. 6b. As seen, the oscillator's response to the ramped-up force shown in Fig. 6a produces no oscillations whatsoever.

The manner in which the force $F(t)$ in Eq. (37) depends on the percentage of critical damping ζ is shown in Fig. 7a. The corresponding response of the oscillator for each of these values of ζ is shown in Fig. 7b. What is significant in all these responses is that there are no overshoots and/or undershoots; the responses monotonically increases without a ripple. We note that when $\zeta = 0$ we recover the ramp function given in Eq. (2) that is linear in time since the coefficients $C_0 = D_0 = 0$ in Eq. (37). This is because in Eq. (34) $\det(A_0) \neq 0$ and now $r_1 = r_2 = 0$, so that $c_0 = d_0 = 0$. Also, when $\zeta \ll 1$ the ramp-up to the final force $F_0 = kx_\mathrm{f}$ is close to linear, as was argued earlier using our continuity argument. But when $0.1 < \zeta < 1$, the force $F(t)$ can be significantly nonlinear in time in the interval $[0, T_n]$ as seen on Fig. 7a. Thus, these results generalize our result in Sect. 2 to include oscillators that are underdamped. We note that the force $F(t)$ can exceed the value F_0 which is required to maintain the desired final constant displacement x_f, as seen in Fig. 7a when $\zeta = 0.8$.

Were we to set $\bar{T} = 2\pi/\omega_d = T_n/\sqrt{1-\zeta^2} := T_d$ in Eq. (35), so that the ramp-up duration is the damped period of vibration of the oscillator, we would get the results shown in Fig. 8 for various values of $0 \leq \zeta < 1$. Note that T_d (duration over which the force is non-constant) now increases as ζ approaches unity, more easily seen in Fig. 8b, which may be a drawback if such a system is required to reach its desired final constant displacement x_f quickly. Yet, the force $F(t)$ ramps up monotonically with $F(T_d) = F_0$, and its magnitude does not exceed $F_0 = kx_\mathrm{f}$. We note that when $\zeta = 0$ the oscillator is undamped and $T_n = T_d$; we then recover the ramp function given in Eq. (2) that is linear in time. The response of the system corresponding to values of ζ considered in Fig. 8a is shown in Fig. 8b. This sort of correspondence will be used in all the plots that follow. As seen, the response increases monotonically to the desired final displacement x_f with no overshoots/undershoots.

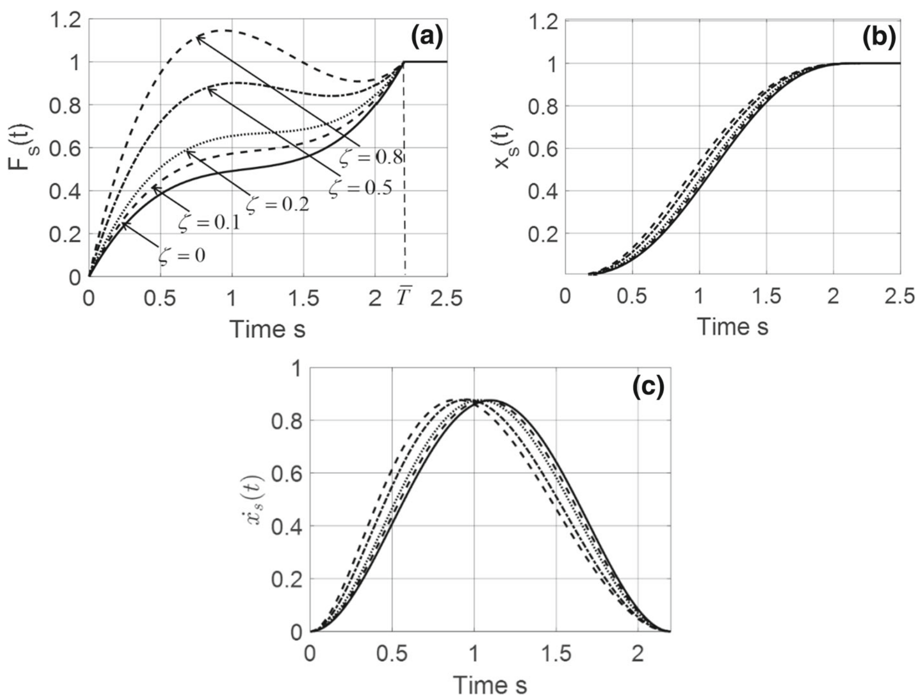

Fig. 9 $m = 2$, $k = 8$, $\bar{T}/T_n = 0.7$. For various ζ values **a** $F_s(t)$, **b** $x_s(t)$, and **c** $\dot{x}_s(t)$

Most SDOF systems met in aerospace, mechanical, and civil engineering have values of $\zeta \le 0.4$, and in many situations, such as mechanical components and the lower modes of vibration in tall buildings, values of $\zeta \ll 1$ are encountered, so that $T_n \approx T_d$. Then, the simple force $F(t)$ given by Eq. (35) with $\bar{T} = T_d$ obtained hereto allows the oscillator's response $x(t)$ to reach its desired final constant value x_f at time T_d with no oscillations and it remains at this constant value thereafter.

However, as pointed out before, there is a time delay T_d in reaching the desired final displacement value x_f, and this delay becomes significant as ζ approaches 1. This can be seen most distinctly in the plot in Fig. 8b of the response $x_s(t)$ that corresponds to the value of $\zeta = 0.8$. If this time delay T_d is acceptable in a given practical situation, then this simple force design would be useful. The acceptability of such a time delay would depend on the damped period of the oscillator, and various practical considerations, such as, how quickly one wants to reach the final desired constant displacement value x_f, and the capabilities of (and constraints on) the actuators required to produce the necessary force.

When ζ is close to unity or when one wants to reach the desire final displacement faster, one could consider using a suitably desired, and acceptable, value of the time $\bar{T} \le T_n$ at which the final displacement value x_f is required to be reached. Figure 9a shows the scaled force $F_s(t)$ needed for the oscillator over the interval $[0, \bar{T}]$ when $\bar{T} = 0.7T_n \approx 2.19$ in the example considered before (with $m = 2$, $k = 8$, so that $T_n = \pi$ s) for various values of the ζ. Comparing with Fig. 8a, we note that the force $F_s(t)$ over the interval $[0, \bar{T} = 2.19]$ is more complex and it is not monotone increasing with time for all ζ values. From a practical standpoint, such a force–time history could perhaps pose greater difficulties in its implementation. The force could well exceed $F_0 = x_f k$, which is needed to achieve the desired final constant displacement value x_f, as shown in Fig. 9a for $\zeta = 0.8$. In fact, when $\bar{T}/T_n \ll 1$, $F(t)$ is oscillatory in the interval $[0, \bar{T}]$ and can have large amplitudes, as is shown later on. However, when $\bar{T}/T_n \ll 1$, though the force $F(t)$ can have large amplitudes and can have oscillatory behavior in the interval $[0, \bar{T}]$, the response $x(t)$ of the oscillator remains smooth and monotonically increases; it has no oscillations (or ripples) whatsoever as seen in Fig. 9b. As shown in Fig. 9c, the velocity $\dot{x}_s(t)$ for each value of ζ is always positive in the interval $[0, \bar{T}]$ (except at $t = 0$ and $t = \bar{T}$, at which it is zero), attesting to the monotonicity of the response, $x(t)$.

Figures 10a, b show the (normalized) response $x_s(t)$ of the oscillator (with $m = 2$, $k = 8$ so that $T_n = \pi s$) and the requisite force $F_s(t)$ for various ζ values shown in Fig. 9a, when $\bar{T}/T_n = 0.015$. Figure 10c shows that in the interval $[0, \bar{T} = 0.0471]$ the system's response $x_s(t)$ does not have a single ripple since its slope, $\dot{x}(t)$, is always positive except at $t = 0$ and $t = \bar{T}$ at which times it is zero.

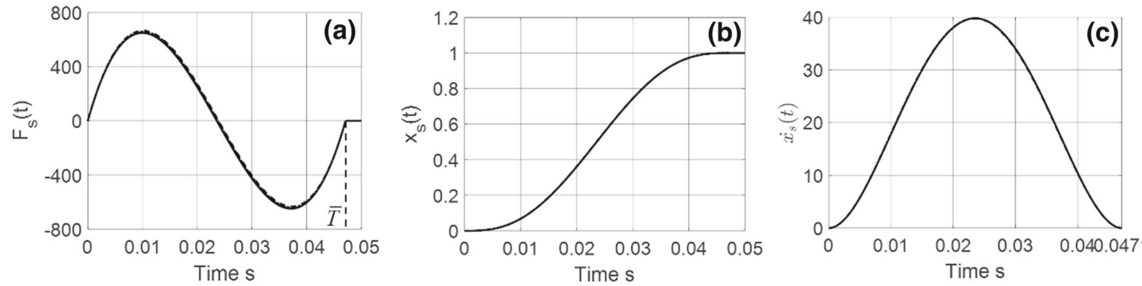

Fig. 10 $m = 2$, $k = 8$, $\bar{T}/T_n = 0.015$. For various ζ values, **a** $F_s(t)$, **b** $x_s(t)$, and **c** $\dot{x}_s(t)$

As stated above, when $\bar{T}/T_n \ll 1$, the force $F_s(t)$ required becomes large in the interval $[0,\ \bar{T}]$, and as shown, it does not seem to change greatly with the value of ζ; the response too is insensitive to the value of ζ. Also $\dot{x}_s(t) \geq 0$ in the interval $t \in [0, \bar{T}]$, showing that $x_s(t)$ monotonically increases in it.

Going back to Eqs. (26) and (27), we have shown that with a proper choice of the parameters b_0, c_0, and d_0 we are assured that for $\tau \geq T$ the oscillator's response $x(\tau)$ to the ramp force $f(\tau)$ remains constant at the desired constant displacement x_{f} with no oscillations for $0 \leq \zeta < 1$. Figures 7b, 8b, 9b and 10b illustrate that for $\tau \in (0, T)$ (i) the response $x(t)$ is smooth and non-oscillatory (in fact, monotone increasing) as seen in Fig. 10b, and (ii) the velocity $\dot{x}(t) \geq 0$, which confirms that the response is monotone increasing, as seen in Fig. 10c.

In summary, the approach is as follows. Consider the system described by Eqs. (26) and (27). Having decided on the value of T which describes the interval $[0, T]$ of time at the end of which the oscillator is desired to reach its desired final displacement value x_{f}, the coefficients $b_0(\zeta, T)$, $c_0(\zeta, T)$, and $d_0(\zeta, T)$ for a given value of x_{f} are obtained from Eqs. (34) and (28); this explicitly gives the force $f(\tau)$. Equation (27) is easily solved for any value of $0 \leq \zeta < 1$, and its solution $x(\tau; \zeta, T, x_{\mathrm{f}})$ is explicitly obtained for $\tau \in [0, T]$ in Appendix 1. This provides $x'(\tau)$. We find that $x'(\tau) \geq 0$ for $\tau \in [0, T]$, when $0 < T \leq T_N = 2\pi$. The response of the underdamped oscillator for $\tau \geq 0$ is non-oscillatory in the interval $\tau \in [0, T]$ as long as $0 < T \leq T_N = 2\pi$ in Eqs. (26) and (27). The force $f(\tau)$ for $\tau \geq T$ is simply $f_0 = x_{\mathrm{f}}$. We have therefore devised a simple methodology to obtain a force $f(\tau)$ that gets any underdamped oscillator from $x(0) = x'(0) = 0$ to a desired final constant displacement x_{f} without a single ripple in its response.

There are both drawbacks and advantages of the simplistic approach that has been developed in this subsection. First, the drawbacks. (1) As shown in Fig. 10, the force becomes very large when the interval of time $[0, T]$ at the end of which the oscillator is required to have the desired final displacement x_{f} is very small when compared to the natural frequency of the oscillator ($T/T_N = \bar{T}/T_n << 1$). As we shall see below, this appears to be the nature of the problem and happens even with the use of more sophisticated methods, as shown below. (2) The approach gives so-called 'open-loop' control, which is not very robust, and is, in general, sensitive to perturbations.

Next, the advantages. (1) The approach to steering the oscillator so that it does not have a single ripple (overshoot/undershoot) in its response presented in this subsection is very simple, and gives closed form results that can be readily used in real-time. It relies solely on the elementary theory of linear vibrations, well within the scope of an introductory course on the subject. (2) Having produced a force–time history for which the response of the system has no oscillations, we have shown that the state described at time T ($\bar{T}$) is 'reachable' from the rest state at time $\tau = 0$ ($t = 0$) while satisfying the necessary and sufficient conditions laid down in Eqs. (24) [or (25)] along with the condition that $f(\tau = 0) = 0$. (3) We have shown that a suitable control force over the interval $[0, T]$ (or, $[0, \bar{T}]$) can be found to satisfy Eq. (24) for all $T > 0$, so that the underdamped oscillator's response has no overshoots/undershoots when $T/T_N \leq 1$.

The approach developed in this section is just one among many. We next explore the use of a different approach to determining the requisite control force over the interval $[0, T]$ (or, $[0, \bar{T}]$) that satisfies the conditions given in Eqs. (24) (or (25)) along with the condition $f(\tau = 0) = 0$ (or $F(t = 0) = 0$).

4.2 Open-loop optimal control

The connection between the two alternative formulations described by Eqs. (26)–(27) and (35)–(36), and the way of using the former to arrive at the latter has been amply described. In what follows, the former formulation will be used in which the time $\tau = \omega_n t$ is dimensionless, and the natural period of the oscillator is normalized to $T_N = 2\pi$.

Instead of approaching the problem from a purely 'vibrations' (dynamics) viewpoint, as was done in the previous subsection, in this subsection we approach the problem from a 'controls' viewpoint. The force $f(\tau)$ is explicitly viewed as a control force, and optimal control is sought.

Were it to be required that the undamped/underdamped oscillator simply be steered from the state $\{x(\tau = 0) = x'(\tau = 0) = 0\}$ to the state $\{x(\tau = T) = x'(\tau = T) = 0\}$ in the interval $[0, T]$, $0 < T \leq T_N = 2\pi$, finding the optimal control force $f(\tau)$ to do this would constitute a standard 'terminal state' optimal control problem [12,13]. However, for the response of the system for $\tau \geq T$ to be free of any oscillations and maintain the desired final displacement x_f, we also require to include the constraint $f(T) = x_\mathrm{f}$ on the control force (see Eq. (24)); additionally, knowing that in most mechanical systems we cannot exert forces instantaneously, we also enforce the constraint $f(0) = 0$.

With these two constraints on the control $f(\tau)$, which are on a set of measure zero and which need to be *exactly* satisfied, standard optimal control formulations cannot be directly applied. This may account for the reason why in the dynamics and controls literature this problem appears not to have been attempted using optimal control theory. One approach to include these constraints on the desired control force $f(\tau)$ would be to compute the optimal trajectory (using the Hamilton–Jacobi approach) without these constraints and then modify it afterwards to satisfy these two force constraints, which are over the two sets of measure zero (at $\tau = 0$ and $\tau = T$). This route, if pursued, appears to encounter subtle aspects of functional analysis, and ends up becoming intractable. In what follows we take a somewhat novel approach, which deals with modifying the very description of the dynamical system itself, albeit preserving its integrity. The approach benefits from being both simpler and more direct.

The second-order system (see Eqs. (26)–(27)) that describes the oscillator with (normalized) natural period $T_N = 2\pi$ over the time interval $\tau \in [0, T]$ is given by (τ is the dimensionless time)

$$x' = y, \; y' = -2\zeta y - x + f(\tau). \tag{38}$$

Assuming that all functions of τ are sufficiently differentiable in the interval $[0, T]$, we consider instead the augmented system in which we *include* the control force $f(\tau)$ as *an element of the state vector* denoting it, for convenience, by the variable $z(\tau) := f(\tau)$, so that Eq. (38) now reads

$$x' = y, \; y' = -2\zeta y - x + z, \quad 0 \leq \tau \leq T. \tag{39}$$

To these two equations we add the 'control' equation

$$z' = w(\tau), \quad 0 \leq \tau \leq T, \tag{40}$$

where $w(\tau)$ is now our 'pseudo-control.' We hence arrive at our 'augmented' system, which is described by Eqs. (39) and (40), which now has the three-dimensional state vector

$$\hat{x}(\tau) = [x(\tau), y(\tau), z(\tau) = f(\tau)]^\mathrm{T}, 0 \leq \tau \leq T, \tag{41}$$

where superscript T denotes the transpose of the row-vector, and our the pseudo-control is $w(\tau)$.

We need to find a suitable (pseudo) control $w(\tau)$ so that the system is steered from its initial state

$$X_i := \{x(0) = 0, \; y(0) = 0, \; z(0) = f(0) = 0\} \tag{42}$$

to its final state

$$X_\mathrm{f} := \{x(T) = x_\mathrm{f}, \; y(T) = 0, \; z(T) = f(T) = x_\mathrm{f}\}. \tag{43}$$

This simple idea of including the actual control force $f(\tau)$ as part of the state equation that describes the augmented system lifts the problem to a 'terminal state' optimal control problem.

Consider the simple cost function J_1 (one could use other cost functions that also penalize large values of $x(t)$ and $y(t)$ for instance) given by

$$J_1 = \frac{1}{2}\int_0^T \left[\alpha f(\tau)^2 + \beta w(\tau)^2\right] \mathrm{d}\tau = \frac{1}{2}\int_0^T \left[\alpha z^2 + \beta w^2\right] \mathrm{d}\tau, \quad \alpha \geq 0, \; \beta > 0 \tag{44}$$

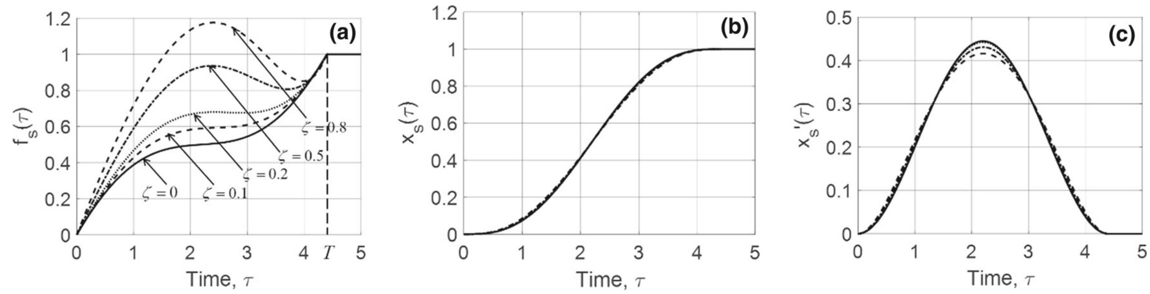

Fig. 11 $\alpha = 0,\ T/T_N = 0.7$. **a** $f_s(\tau)$, **b** $x_s(\tau)$, **c** $x_s'(\tau)$, for different ζ values

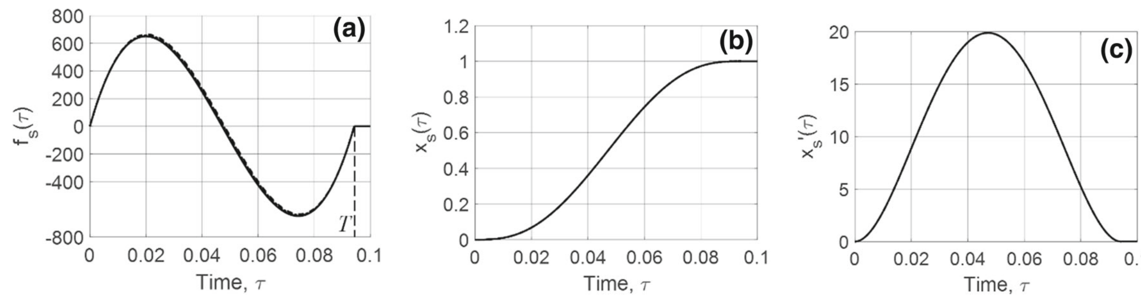

Fig. 12 $\alpha = 0,\ T/T_N = 0.015$. **a** $f_s(\tau)$, **b** $x_s(\tau)$, **c** $x_s'(\tau)$, for different ζ values

where α and β are suitable constants. Note that the second member under the integral in Eq. (44) penalizes excessive *changes* in the actual control force $f(\tau)$ since $w(\tau) = f'(\tau) = z'(\tau)$.

Using the Lagrange multipliers $\lambda_i(\tau),\ i = 1, 2, 3$, we extremize the functional

$$\hat{J}_1 = J_1 + \int_0^T [\lambda_1(\tau)(y - x') + \lambda_2(\tau)(-2\zeta y - x + z - y') + \lambda_3(\tau)(w - z')]\mathrm{d}\tau \tag{45}$$

by using the calculus of variations. This yields the following state and costate equations (see Appendix 2) over the time interval $\tau \in [0, T]$

$$\begin{aligned} x' &= y, & \lambda_1' &= \lambda_2, \\ y' &= -2\zeta y - x + z, & \lambda_2' &= 2\zeta\lambda_2 - \lambda_1, \\ z' &= w, & \lambda_3' &= -\lambda_2 - \alpha z, \quad \text{and} \quad w = -\tfrac{\lambda_3}{\beta}, \end{aligned} \tag{46}$$

subject to the boundary conditions X_i and X_f given in Eqs. (42) and (43). Its solution provides the optimal 'pseudo-control' $w(\tau)$, and ensures that the actual control force $f(\tau) := z(\tau)$ satisfies the constraints at the endpoints of the interval. Somewhat surprisingly, a closed form solution to these equations can be determined. It is easy to show formally that our augmented system given by Eqs. (39) and (40) is reachable, a conclusion that also follows from considerations of mechanics.

Figure 11 shows the steering force, and the response of the system with $\alpha = 0$, and $T/T_N = T/2\pi = 0.7,\ T \approx 4.4$, for various values of ζ. Since $\alpha = 0$, no cost (see Eq. (44)) is attached to the magnitude of the steering force $f(\tau)$, and the value of β becomes inconsequential. The plots shown are scaled so that $x_s(\tau) = x(\tau)/x_\mathrm{f}$, and $f_s(\tau) = f(\tau)/x_\mathrm{f}$. As seen in Fig. 11c, $x_s'(\tau) \geq 0$ over the interval $[0, T]$ indicating that the response has no ripple or oscillations whatsoever.

Figure 12 shows the steering control force and the response when $\alpha = 0, T/T_N = 0.015$. Again the response is smooth and does not have any oscillations (ripples), though the time interval $[0, T \approx 0.094]$ during which the oscillator reaches its final desired constant value is now 0.015 of its natural period, T_N. The various curves in Fig. 12 for different ζ values shown in Fig. 11a almost fall on each other and are indistinguishable at the scale shown; the percentage of critical damping appears to have little effect on either the steering force $f_s(\tau)$, which is large and oscillatory now, and on the oscillator's response.

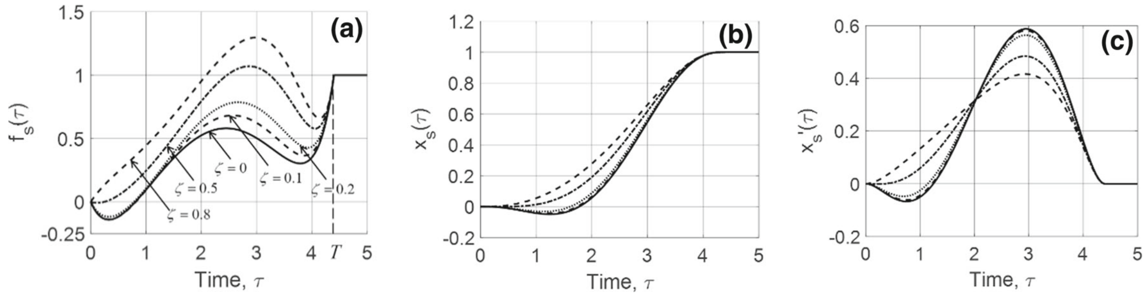

Fig. 13 $\alpha = 10$, $\beta = 1$, $T/T_N = 0.7$. **a** $f_s(\tau)$, **b** $x_s(\tau)$, **c** $x_s'(\tau)$, for different ζ values

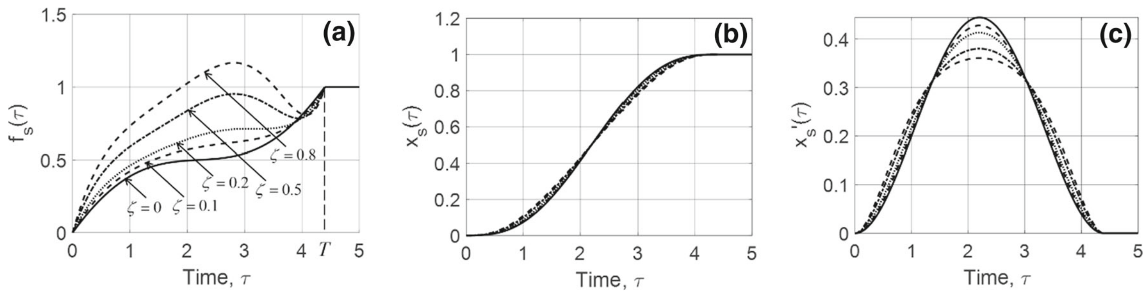

Fig. 14 $\alpha = 10$, $\beta = 1$, $T/T_N = 0.7$. **a** $f_s(\tau)$, **b** $x_s(\tau)$, **c** $x_s'(\tau)$, for different ζ values

Were we to include in our cost function a cost associated with the magnitude of the steering force $f(\tau) := z(\tau)$, by choosing a value of $\alpha > 0$ (see Eq. (44)), the response of the oscillator turns out to be oscillatory in the interval $[0, T]$ as shown in Fig. 13. Taking $\alpha = 10$, $\beta = 1$, the results show that the oscillator's response is now no longer always a monotonic increasing function of time for all values of ζ as shown in Fig. 13b, and more clearly so in Fig. 13c where we see that the velocity $x_s'(\tau)$, is continuous and changes sign. For smaller values of ζ the transient response is non-monotone increasing in time. The steering control force also does not monotonically increase.

While the cost function is often taken by control theorists to minimize the L_2 norm of the control force often for purposes of analytical tractability, from a physical standpoint it appears that a better criterion for us here might be the minimization of the work done by the control force $f(\tau)$ over the interval $[0, T]$.

Using the cost function J_2

$$J_2 = \frac{1}{2}\int_0^T \left[\alpha f(\tau)y(\tau) + \beta w(\tau)^2\right] \mathrm{d}\tau = \frac{1}{2}\int_0^T \left[\alpha z(\tau)y(\tau) + \beta w^2\right] \mathrm{d}\tau, \quad \alpha \geq 0,\ \ \beta > 0, \tag{47}$$

in which the first term now places a cost on the work done by the steering force in the interval $[0, T]$, one obtains, after using the augmented cost function as in Eq. (45) with the Lagrange multipliers, the state and costate equations as

$$\begin{aligned} x' &= y, & \lambda_1' &= \lambda_2, \\ y' &= -2\zeta y - x + z, & \lambda_2' &= 2\zeta\lambda_2 - \lambda_1 - \tfrac{\alpha}{2}z, \\ z' &= w, & \lambda_3' &= -\lambda_2 - \tfrac{\alpha}{2}y, \quad \text{and} \quad w = -\tfrac{\lambda_3}{\beta}. \end{aligned} \tag{48}$$

The derivation of the costate equations follow in a manner similar to that shown in Appendix 2, and is omitted here for brevity. An analytical solution to Eq. (48) subject to the boundary conditions (42) and (43) is difficult, and they are numerically solved using Matlab's bvp4c function. The two cost functions J_1 and J_2 are identical when $\alpha = 0$. However when $\alpha > 0$, the cost function (47) yields a monotonically increasing response $x(\tau)$ in the interval $[0, T]$ with no oscillatory behavior.

The next two figures (Figs. 14, 15) relate to results obtained using the cost function J_2 that aims to minimize the work done by the control force $f(\tau)$ over the interval $[0, T]$.

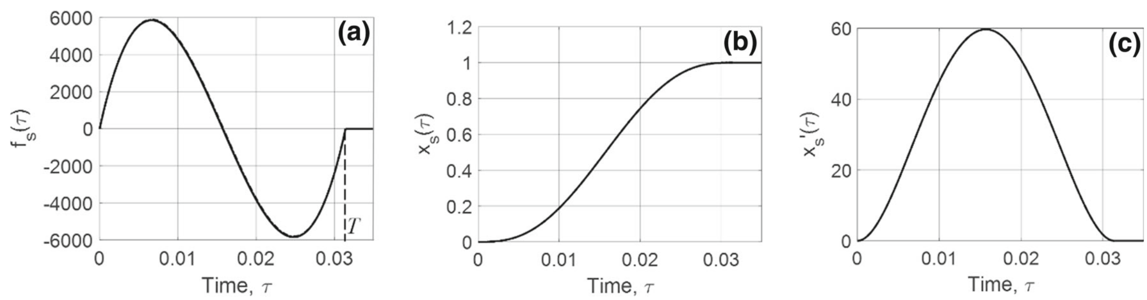

Fig. 15 $\alpha = 10, \beta = 1, T/T_N = 0.005$. **a** $f_s(\tau)$, **b** $x_s(\tau)$, **c** $x_s'(\tau)$, for different ζ values

Figure 14 shows the optimal control force $f_s(\tau) = f(\tau)/x_{\mathrm{f}}$ for various values of ζ when the duration of time T set to reach the final desired displacement is $T = 0.7T_N \approx 4.4$. Note that the response $x_s(t)$ shown in Fig. 14b over the interval $[0, T]$ is monotonically increasing. Furthermore, as seen from the slope of this curve shown in Fig. 14c, which is positive except at $\tau = 0$ and at $\tau \geq T$, the response $x_s(t)$ has no oscillations whatsoever.

Figure 15 shows the steering control for $T/T_N = 0.005$. In Fig. 15b we observe the smooth response $x_s(t)$ of the oscillator, with which the desired final constant value of displacement is reached, starting from rest. This final displacement is reached at the end of 1/200th of the oscillator's dimensionless frequency $T_N = 2\pi$. This is done of course at the expense of a very large-amplitude force $f_s(\tau) = f(\tau)/x_{\mathrm{f}}$ as seen in Fig. 15a. The various curves for the different values of ζ shown in Fig. 14a fall closely on top of one another as shown in Figs. 15b, c.

Having used optimal control, which works well and provides the facility to use different cost functions and weight the control cost as necessary, the control still lacks robustness. The pseudo-control force is not a function of the current state $\hat{x}(\tau)$ (see Eq. (41)); it can in fact be precomputed, and then applied for τ belonging to the interval $[0, T]$. If, for some reason, the state is perturbed off the optimal trajectory, then such an open-loop control will not satisfy the conditions given in Eq. (43) at time $\tau = T$. Both the control approaches dealt with so far have this disadvantage. We therefore next seek a closed-loop feedback pseudo-control force $w(\tau)$, which would substantially reduce the sensitivity of the control to deviations that could arise during practical implementation.

4.3 Closed-loop optimal control

Using again the augmented dynamical system with the (pseudo) control $w(\tau)$ described by Eqs. (39) and (40), namely (recall, $z(\tau) \equiv f(\tau)$)

$$\dot{\hat{x}} := \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ -1 & -2\zeta & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{\hat{A}} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w, \quad 0 \leq \tau \leq T, \tag{49}$$

we determine $w(\tau)$ in this subsection by considering closed-loop optimal control (CLOC). CLOC is obtained for the 'terminal control' problem with boundary conditions given by Eqs. (42) and (43). That is, by using a suitable closed-loop pseudo-control input, $w(\tau)$, $0 \leq \tau \leq T$, the aim is to steer the (augmented) state vector $\hat{x}(\tau)$ from its initial value $\hat{x}(0) = 0$ to its terminal value $\hat{x}(T) = [x_{\mathrm{f}}, 0, x_{\mathrm{f}}]^{\mathrm{T}}$.

Consider the cost function [13]

$$J_3 = \frac{1}{2}\hat{x}^T(T)S(T)\hat{x}(T) + \frac{1}{2}\int_0^T \left[\hat{x}^T Q\hat{x} + \beta w^2\right] \mathrm{d}t, \tag{50}$$

in which the 3 by 3 matrices $S(T) \geq 0$, $Q \geq 0$, and the scalar $\beta > 0$.

We denote the symmetric matrix

$$S(\tau)=\begin{bmatrix} s_1(\tau) & s_2(\tau) & s_3(\tau)\\ s_2(\tau) & s_4(\tau) & s_5(\tau)\\ s_3(\tau) & s_5(\tau) & s_6(\tau)\end{bmatrix}\geq 0,\quad \text{and}\quad S_1(\tau)=[s_3,\ s_5,\ s_6] \tag{51}$$

and the 3 by 3 matrix

$$V(\tau)=[v_{i,j}(\tau)]\quad \text{with}\quad V_1(\tau)=[v_{31},v_{32},v_{33}]. \tag{52}$$

The following two matrix equations are then simultaneously solved backwards [12]:

$$-\dot{S}=\hat{A}^T S+S\hat{A}-\frac{1}{\beta}\begin{bmatrix} s_3\,S_1\\ s_5\,S_1\\ s_6\,S_1\end{bmatrix}+Q,\quad 0\leq\tau\leq T,\ \text{given } S(T) \tag{53}$$

and

$$\dot{V}=\begin{bmatrix} 0 & 1 & s_3/\beta\\ -1 & 2\zeta & s_5/\beta\\ 0 & -1 & s_6/\beta\end{bmatrix}V,\quad 0\leq\tau\leq T,\ V=I. \tag{54}$$

In addition to the integral term in Eq. (50), the matrix $S(T)$ provides, in general, the facility of penalizing deviations of the terminal state from zero so that the feed-back optimal control reduces such deviation at $\tau=T$. The only component of the terminal state that is needed to be brought to zero is y. Hence, if desired, in order to bring $y(T)$ closer to zero, one could specify in the computations $S(T)=b_e\mathrm{Diag}(0,1,0)$ by using a suitable constant value of $b_e>0$.

From the solution of Eqs. (53) and (54), the symmetric matrix

$$P(\tau)=\int_T^{\tau}\frac{1}{\beta}\begin{bmatrix} v_{31}(s)\,V_1(s)\\ v_{32}(s)\,V_1(s)\\ v_{33}(s)\,V_1(s)\end{bmatrix}\mathrm{d}s \tag{55}$$

is obtained [12]. The optimal feedback control is then obtained as

$$w(\tau)=-\frac{1}{\beta}\left[s_3\ s_5\ s_6\right]\hat{x}(\tau)+\frac{1}{\beta}\left[v_{31}\quad v_{32}\quad v_{33}\right]P^{-1}(\tau)\left\{V^T\hat{x}(\tau)-\hat{x}(T)\right\},\quad 0\leq\tau\leq T. \tag{56}$$

It should be pointed out that this feed-back control, $w(\tau)$, aims to have the augmented system *exactly* reach its terminal state; it is then used to determine the response of the augmented system described by Eq. (49). The actual control $f(\tau)$ applied to the oscillator (see Eq. (26)) is of course the third component, $z(\tau)$, of the augmented state vector, $\hat{x}(\tau)$, which can be obtained from $\hat{x}(\tau)$.

The first member on the right-hand side of Eq. (50) could also be set to zero by setting $S(T)=0$ (or, $b_e=0$). Further, when $Q=0$ solving Eq. (53) backwards with this boundary condition would obviously result in the solution $S(\tau)\equiv 0$ for $0\leq\tau\leq T$, and consequently the first member on the right hand side of Eq. (56) becomes zero. The feedback pseudo-control $w(\tau)$ is then proportional only to $\{V^T\hat{x}(\tau)-\hat{x}(T)\}$, and it is independent of the parameter β.

We note from Eq. (55) when $\tau=T$, $P(T)=0$, and so $P^{-1}(T)$ does not exist. As is well known [12], the solution to the terminal optimal control problem has the drawback that the control does not remain finite at the endpoint $\tau=T$ of the interval $[0,T]$. In fact, this is the price that is paid for requiring that the control exactly drive the (augmented) system to its terminal state, $\hat{x}(T)$.

To circumvent this singularity problem at $\tau=T$ which arises in terminal state feedback optimal control problems, in this paper the following is done. The integration of the augmented system (Eq. (49)) with the feedback control $w(\tau)$ determined in Eq. (56) is carried out for a time slightly less than T (i.e., before the singularity is reached), namely for a time $T_\gamma=(1-\gamma)T,\ \gamma<<1$. This gives us the augmented state vector, $\hat{x}(\tau)$, in the interval $[0,T_\gamma]$, and thus the value of $z(T_\gamma)$, and therefore $f(T_\gamma)$, since $z(\tau)=f(\tau)$ (see Eq. (26)). Over the short interval $[(1-\gamma)T,T]$, Eq. (26) is used to determine the response of the oscillator, and

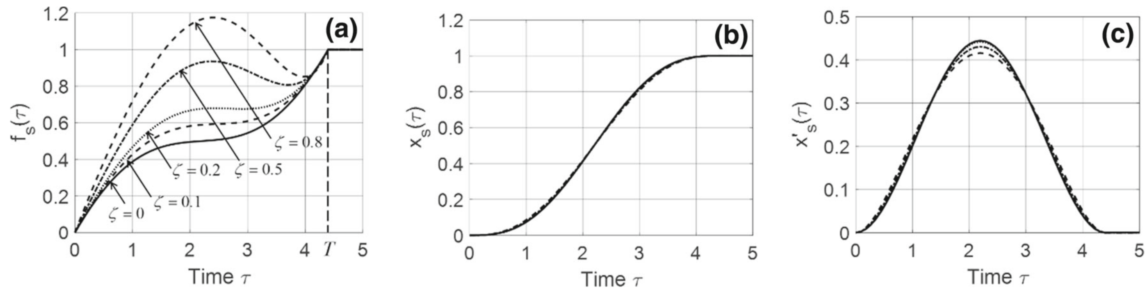

Fig. 16 $S(T) = Q(T) = 0,\ T/T_N = 0.7$. **a** $f_s(\tau)$, **b** $x_s(\tau)$, **c** $x_s'(\tau)$, for different ζ values

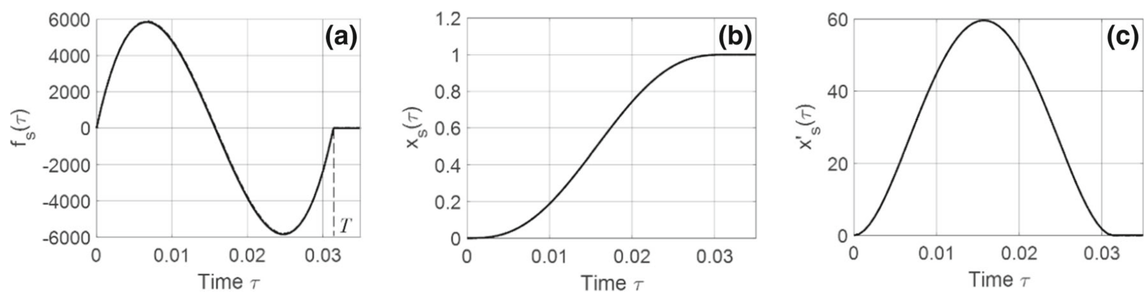

Fig. 17 $\alpha = 10, \beta = 1, S(T) = 0, T/T_N = 0.0057$. **a** $f_s(\tau)$, **b** $x_s(\tau)$, **c** $x_s'(\tau)$, for different ζ values

the force $f(\tau)$ in this interval is obtained by linearly interpolating $f(\tau)$ between its values $f(T_\gamma) = z(T_\gamma)$ and $f(T) = x_{\mathrm{f}}$ at the two ends of this interval, so that

$$f(\tau) = f(T_\gamma) + [x_{\mathrm{f}} - f(T_\gamma)]\frac{\tau - T_\gamma}{T - T_\gamma}, \quad T_\gamma = (1-\gamma)T \le \tau \le T. \tag{57}$$

Through approximate, this control force in the interval $[(1-\gamma)T, T]$ appears to work reasonably well. One could use a very small value of γ and reduce this interval to become minute by using higher precision arithmetic in the computations.

Some numerical results from using the feedback optimal control approach are shown in Fig. 16. With $S(T) = Q = 0$ and $\gamma = 1\mathrm{e}{-3}$, the results for $T = 0.7T_N$ are shown for different values of ζ.

Comparing this Figure with Fig. 11, we see that we have obtained the feed-back optimal control version now for the open-loop control that was obtained earlier in Fig. 11. The control forces $f_s(\tau)$ in the two figures are the same, as are the responses shown in Fig. 16b for the various values of ζ. Note that there is not a single ripple in the response, as seen from Fig. 16c which shows that the slope, $x'(\tau)$, is always positive except at $\tau = 0$ and $\tau \ge T$ where is zero.

Similarly were we to take $S(T) = 0$ and $Q = \mathrm{Diag}(0, 0, \alpha)$ so that we are penalizing large values of $f(\tau) = z(\tau)$, we would get, for $\alpha = 10$ and $\beta = 1$, the results shown in Fig. 13, because we would simply obtain the closed-loop version of the open-loop control that we had earlier, and the cost functions J_1 and J_2 would be identical.

Lastly, we show the results of closed-loop optimal control for a different value of $T/T_N = 0.005$ for the five values of ζ shown in Fig. 16a. We take $\alpha = 10$ and $\beta = 1$. The response is smooth and monotone increasing in time with no overshoots or undershoots. As observed before, the response is insensitive to the values of ζ.

There is a striking similarity between Figs. 15 and 17. However, the cost function used to obtain the result in Fig. 15 is J_2 while that used to obtain Fig. 17 is J_3. When the interval of time is very short (compared to the normalized period, $T_N = 2\pi$) over which the desired final displacement is to be reached, the two cost functions give about the same kind of control.

The three approaches used above by no means provide a totality of ways in which the terminal control problem of transferring the state of the (normalized) oscillatory system from its zero (displacement and velocity) state to the required terminal state $x(T) = x_{\mathrm{f}}$ and $x'(T) = 0$, while using a force that also satisfies the terminal conditions $f(0) = 0$ and $f(T) = x_{\mathrm{f}}$. Several variants of these approaches can be used, along with other entirely different methods. The three approaches provided here simply typify three common modes of thought—the

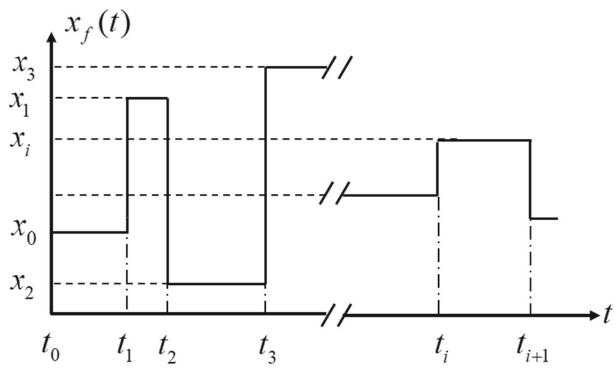

Fig. 18 Desired piecewise constant displacement trajectory $x_f(t)$

idea of using a simplistic force described by a low-order polynomial, the idea of using optimal open-loop control, and then closed-loop optimal control.

5 Piecewise constant displacement time-history tracking for undamped and underdamped system with no overshoots and undershoots

The methods suggested above can now be extended and applied to the problem of tracking a sequence (in time) of constant displacement states $x_f(t)$, as shown in Fig. 18, for an underdamped oscillator. The response is required to have no ripples whatsoever. Such a requirement to track a piecewise constant displacement trajectory is useful in many applications such as power systems that distribute constant power, chemical mixers that require constant amounts of ingredients to be mixed to produce different products, and more importantly in various applications in civil, mechanical, and aerospace engineering, where large instantaneous forces are difficult, if not impossible, to generate.

To illustrate the simplicity of the general approach, consider the desired piecewise constant trajectory $x_f(t)$ shown in Fig. 18 demanded of an underdamped (or undamped) SDOF system. In other words, the SDOF system is required to have a sequence of constant displacement amplitudes given by

$$x_f(t) = x_i, \quad \text{for} \quad t \in (t_i, t_{i+1}], \quad t_{i+1} > t_i, \quad i = 0, 1, 2, \ldots$$

with jumps in the displacement of the oscillator at desired times $t_0, t_1, t_2, \ldots$.

The aim is to find sets of simple practical (control) forces to be provided to the underdamped SDOF system so that it follows this piecewise constant displacement trajectory as closely as possible, without *any* overshoots or undershoots. We argue as follows.

In order to have x_i as the desired final displacement in the interval $(t_i, t_{i+1}]$ we ultimately need a constant force $F_i = kx_i$ in this interval. We assume likewise that in the previous interval $(t_{i-1}, t_i]$ within which the desired constant displacement state is x_{i-1}, the constant force $F_{i-1} = kx_{i-1}$ has ultimately been used. To reach such a constant displacement level without any overshoots or undershoots in the interval $(t_i, t_{i+1}]$ all that is needed is to suitably ramp up (or ramp down), over a desired interval of time $\bar{T} \leq t_{i+1} - t_i$, the (constant) force F_{i-1} used in the previous interval to F_i, which is the force to be ultimately employed in this interval. For no oscillations to occur in the interval $[t_i + \bar{T}, t_{i+1}]$, the state at time t_i must be propagated to the desired state x_i at time $t_i + \bar{T}$ so that the necessary and sufficient conditions (see Eqs. (24) and (25))

$$x(t_i + \bar{T}) = x_i, \dot{x}(t_i + \bar{T}) = 0, \; F(t_i + \bar{T}) = kx_i, \quad i = 0, 1, 2, \ldots. \tag{58}$$

be satisfied.

One can do this by using any of the three elementary methods proposed in the previous section, or perhaps some other suitable methods.

For example, using the low-order polynomial considered in Sect. 4.1, the ramp force function in the interval $[t_i, t_{i+1}]$

$$F_i(t) = \begin{cases} F_{i-1} + B_0\tilde{t} + C_0\tilde{t} + D_0\tilde{t}, & 0 < \tilde{t} \leq \bar{T} \\ k\, x_i, & t_i + \bar{T} \leq t \leq t_{i+1}, \end{cases} \tag{59}$$

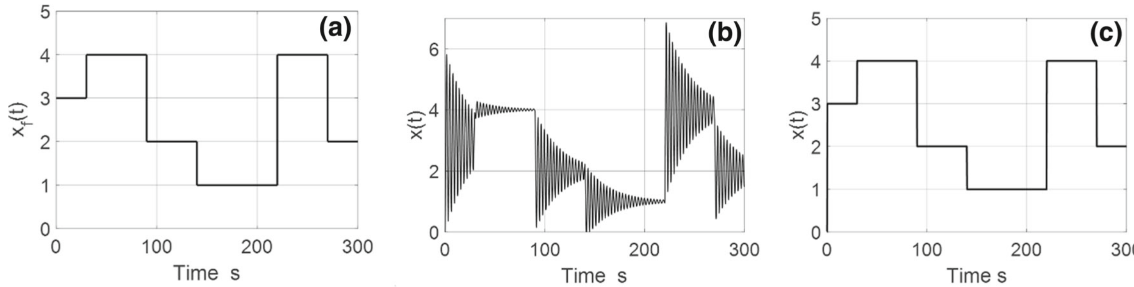

Fig. 19 $m = 2$, $k = 8$, $\zeta = 0.02$. **a** Desired response, $x_f(t)$, **b** step-function response, **c** closed-loop optimal control response $x(t)$

where $\tilde{t} = t - t_i$, and $0 < \bar{T} \leq T_n$, will provide a transition between two levels of constant displacement without a single ripple of oscillation between the two levels. The values of the constants $B_0, C_0,$ and D_0 are obtained as in Sect. 4.1 so that the SDOF system gets a sustained (long-time) displacement response change of $(x_i - x_{i-1})$ over a duration $\bar{T}$ when starting from zero initial conditions. Instead of using the polynomial force over the interval $0 < \tilde{t} \leq \bar{T}$ one could use the force obtained by using open-loop control (Sect. 4.2) or by using closed-loop control (Sect. 4.3).

We next illustrate tracking control of piecewise constant trajectories using the closed-loop optimal control approach described in Sect. 4.3 that ensure the conditions given in Eq. (58). Instead of using the polynomial force function $B_0\tilde{t} + C_0\tilde{t}^2 + D_0\tilde{t}^3$ in Eq. (59), a closed-loop optimal control force $F(\tilde{t})$, which is more robust, is used.

For illustration, we consider an underdamped oscillator ($m = 2$, $k = 8$, $\zeta = 0.02$) that starts from zero initial conditions. Our aim is to track the time-history $x_f(t)$ shown in Fig. 19a. The desired piecewise constant displacement of the oscillator has jumps at times $t_i = 0$, 30, 90, 140, 220, and 270 s.

The oscillator's natural period $T_n = \pi\ s$. Figure 19b shows the displacement response of the oscillator when subjected to a control force comprising a series of step functions of magnitude $F_i(t) = k\,x_f(t_i)$, at times $t = t_i$. As expected, the lightly damped oscillator has large amplitude overshoots and undershoots in its response. Figure 19c shows the response obtained using the closed-loop optimal control approach developed in Sect. 4.3, with $S = Q = 0$ in Eq. (50). Over each of the time intervals, $(t_i, t_i + \bar{T}]$, the force in ramped up (or down). The time over which the force is non-constant in each interval $(t_i, t_{i+1}]$ is $\bar{T} = 0.1T_n \approx 0.314$ s, which is a tenth of the natural period of the oscillator. In each of these intervals, beyond the time $\bar{T}$ the force is a constant and has a value of $k\,x_f(t_i)$, which is the same value as for the series of step functions used to generate the response shown in Fig. 19b. The response of the oscillator to the closed-loop control is shown in Fig. 19c. As seen, the oscillator's response follows the desired piecewise constant trajectory very closely in each time segment $(t_i, t_{i+1}]$, except initially over an interval of time $\bar{T}$. The response of the oscillator to the closed-loop control force shown in Fig. 19c is smooth and has *no overshoots or undershoots*.

One can see this more clearly in Fig. 20, which shows the response on an expanded time scale over two intervals when the amplitude of the oscillator's displacement response changes in what looks like a 'step' in Fig. 19c. Notice how each of these forces levels off at the value of the force required to maintain the constant displacement values (see Eq. (59)) required in each interval over which the displacement is piecewise constant. The response has no overshoots or undershoots.

Lastly, we consider the situation in which the time to transit between two different consecutive values of the desired piecewise constant displacement is required to be very small compared to be the oscillator's period, a situation not altogether uncommon.

Figure 21a shows the same desired piecewise constant trajectory to be tracked by the same oscillator ($m = 2$, $k = 8$, $\zeta = 0.02$) but the timescale in Fig. 19a has been reduced by a factor of 100.

The oscillator, whose natural period $T_n = \pi\ s$, when subjected to the series of step functions $F_i = k\,x_f(t_i)$ applied at time $t = t_i$ that produce the constant steady state response levels shown in Fig. 21c, shows a smooth displacement response (Fig. 21b). Its natural period being $\pi\ s$, the oscillator cannot 'feel' all the 'small' duration changes in the step function force profile $F(t)$ whose shape looks exactly like the one in Fig. 21a, except that the ordinate now is multiplied by k.

Using the closed-loop control that is developed, the trajectory tracking is shown in Fig. 21c. In each of the subintervals $(t_i, t_{i+1}]$ in which the desired displacement is required to be a constant, the time $\bar{T}$ over which the closed-loop control force (and therefore also the displacement response) is non-constant is prescribed to

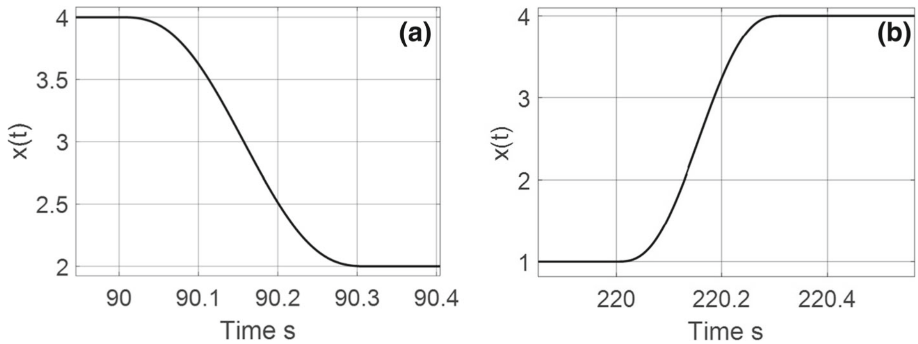

Fig. 20 Response to closed-loop control force $F(t)$ around time **a** $t = 90$ s, **b** $t = 200$ s

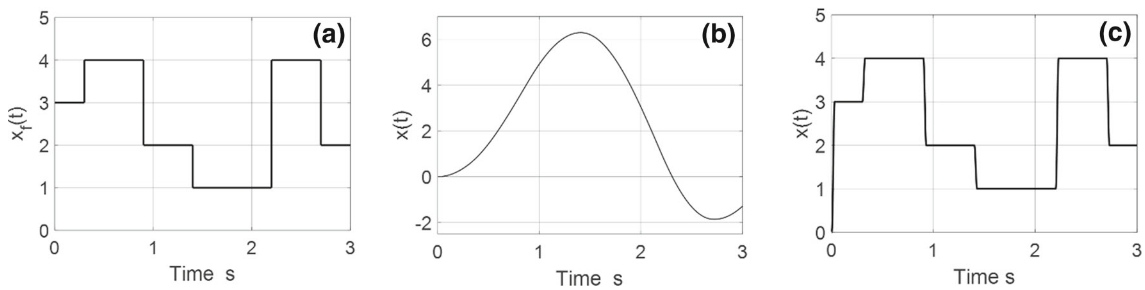

Fig. 21 $m = 2$, $k = 8$, $\zeta = 0.02$. **a** Desired response, $x_f(t)$, **b** step-function response, **c** closed-loop optimal control response $x(t)$

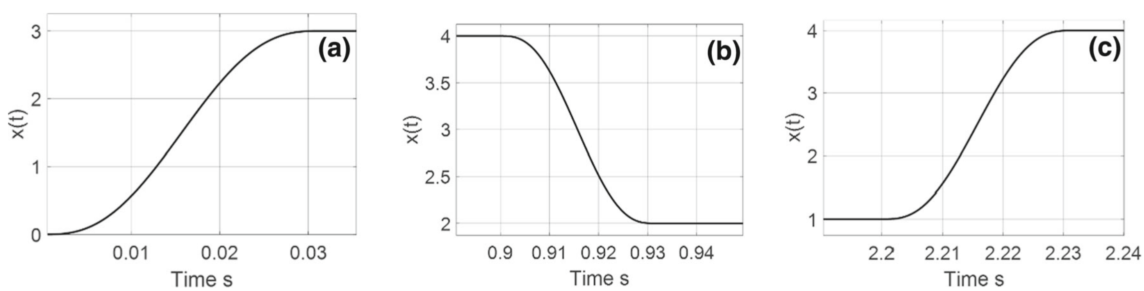

Fig. 22 Response to closed-loop control around time **a** $t = 0$ s, **b** $t = 0.9$ s, **c** $t = 2.2$ s

be $\bar{T} = 0.01T_n \approx 0.0314$ s. The response to the closed-loop optimal control force that tracks the desired displacement trajectory is smooth and has *no* overshoots or undershoots as shown in Fig. 21c.

This is seen more clearly from Fig. 22 where the response is shown at three locations of time where a jump in the desired displacement occurs. As seen, the response of the oscillator is smooth with no oscillations. We thus have a simple closed-loop control approach to have the oscillator closely track a piecewise constant trajectory in a smooth manner with no overshoots or undershoots. To achieve this, the force applied to it in each time interval $(t_i, t_{i=1}]$ is non-constant only over the short sub-interval $(t_i, t_i + \bar{T}]$. The price paid for making $\bar{T}$ very small is, of course, that the force required in these short sub-intervals of duration $\bar{T}$ would need to be large (see Fig. 17).

6 Conclusions

This paper explores a commonly occurring practical problem in aerospace, mechanical, and civil engineering systems in which a system is required to follow a piecewise constant trajectory. Step function forces provide oscillatory responses, with significant overshoots and undershoots, especially for lightly damped systems. From a practical standpoint, because of inertia in a mechanical system it is often very difficult, if not impossible, to generate large instantaneous forces that can be modeled as step functions. The paper provides a general approach to the development of simple, practical force–time histories that can provide the capability to track such piecewise constant displacement trajectories *without* any overshoots and undershoots for an underdamped (and even undamped) single degree of freedom (SDOF) system. While the paper deals with SDOF systems, the approach developed here is also applicable to classically damped multi-degree of freedom systems that have underdamped classical modes of vibration.

The paper first investigates the development of force–time histories $F(t)$ in which the force is increased not instantaneously, but over a suitably small (prescribed) interval of time, say $\bar{T}$, so that an underdamped/undamped oscillator starting from rest attains a desired constant value of displacement x_{f} at the end of this interval and maintains it forever thereafter without there being any undershoots or overshoots in its response.

As a prelude to this, in Sect. 2 the study of an undamped oscillator is considered. It is shown that if (i) the force applied to the undamped system is ramped up linearly in time, (ii) the interval $\bar{T}$ is chosen to be the natural period T_n of the oscillator, and (iii) the force $F(t)$ is made to reach the value needed to maintain the desired constant steady state displacement, i.e., $F(t) = kx_{\mathrm{f}}$ when $t \geq \bar{T}$, where k is the stiffness of the oscillator, the undamped oscillator's response will smoothly increase from its initial value of zero to its desired final constant displacement over the interval $\bar{T}$ and remain there thereafter, with *no* overshoots or undershoots. To the best of the author's knowledge, this simple result, which requires use of just the elementary theory of vibrations appears to have gone unnoticed so far in the literature dealing with the theory of vibrations.

The underlying idea, which serves as the inspiration for the following section, is simple. To have an undamped oscillator (initially at rest with zero initial displacement) reach a constant (nonzero) desired static displacement value, instead of using a step function type of force, which would cause the response to endlessly overshoot and undershoot the desired constant displacement value, one simply employs a ramp force over a short duration of time. After the force reaches the value required to produce the required constant (static) displacement, it is held constant at that value. A linear (in time) ramp over a duration of the natural period of an undamped oscillator followed by a constant force that would generate the desired constant displacement is shown to completely suppress the oscillations of an undamped oscillator when it is required to reach a constant displacement response.

Necessary and sufficient conditions are obtained so that a desired constant displacement is achieved by an uderdamped oscillator (starting from rest with zero displacement) for all time $t \geq \bar{T}$, where $0 < \bar{T} \leq T_n$, where T_n is the natural period of the oscillator. Using these conditions, force–time histories are obtained in which an underdamped oscillator can reach a constant (nonzero) displacement at the end of a pre-specified interval $\bar{T} \leq T_n$ and maintain this displacement thereafter with no overshoots and undershoots (oscillations) in its entire response. A preliminary approach using a polynomial forcing function is first considered, establishing thereby the 'realizability' of such a control force that causes no ripples in the response of the underdamped oscillator and maintains a constant response at and beyond a pre-specified duration of time. Open-loop and closed-loop optimal control are then considered; compared to open-loop optimal control, the latter leads to robust optimal control. It is shown that the development of an optimal control falls outside the standard framework of the so-called terminal control problem because it requires that the control force be constrained over sets of measure zero in time. This makes the Hamilton–Jacobi approach difficult, if not impossible, to use. It appears that the recognition of such constraints may have hereto prevented the use of closed-loop control methods in handling the problem. In this paper a simple new approach is developed to surmount this problem by modifying the very description of the dynamical system, thereby making it amenable to both open- and closed-loop control. A rippleless response is thereby engendered by the oscillator, resulting finally in constant displacement beyond a pre-specified interval of time.

The ease and efficacy with which the closed-loop control is obtained is demonstrated by having a lightly damped oscillator track a piecewise constant time-history, with no overshoots and/or undershoots. This is a common requirement not just in the control of vibrating aerospace, civil, and mechanical systems but in many others; for example, in the power industry when piecewise constant power levels may be required, and in process/food industries where piecewise constant levels of chemicals/ingredients are desired to be mixed together and/or produced.

This work represents only the beginning of an exploration into determining simple, robust force time histories to achieve commonly desired response time histories from underdamped/undamped linear dynamical systems, from a vibrations (or mechanics) viewpoint and, correspondingly, also from a controls perspective.

Appendix 1

Consider the equation

$$x'' + 2\zeta x' + x = f(t),\;\; x(0) = 0,\;\; x'(0) = 0,\quad 0 \le \tau \le T, \tag{60}$$

in which $f(\tau) = b_0\tau + c_0\tau^2 + d_0\tau^3$.

Since we require that $f(T) = x_\mathrm{f}$,

$$b_0 = \frac{xf}{T} - c_0 T - d_0 T^2 \tag{61}$$

so that

$$f(\tau) = \left(-T\tau + \tau^2\right) c_0 + \left(-T^2\tau + \tau^3\right) d_0 + \frac{xf\tau}{T}. \tag{62}$$

The solution to Eq. (60) with this right hand side is

$$x(\tau) = A + B\tau + C\tau^2 + D\tau^3 + \exp(-\zeta\tau)[H_1\cos(S\tau) + H_2\sin(S\tau)], \tag{63}$$

where $S = \sqrt{1-\zeta^2}$, $0 \le \zeta < 1$. Its derivative with respect to τ is

$$x'(\tau) = B + 2C\tau + 3D\tau^2 + \exp(-\zeta\tau)[(SH_2 - \zeta H_1)\cos(S\tau) - (SH_1 + \zeta H_2)\sin(S\tau)]. \tag{64}$$

The constants in Eqs. (63) and (64) are

$$A = 2\left(T\zeta + 4\zeta^2 - 1\right) c_0 + 2\left(T^2\zeta - 24\zeta^3 + 12\zeta\right) d_0 - \frac{2\zeta x_\mathrm{f}}{T}, \tag{65}$$

$$B = -(T + 4\zeta)\, c_0 - \left(T^2 - 24\zeta^2 + 6\right) d_0 + \frac{x_\mathrm{f}}{T},\;\; C = c_0 - 6\zeta d_0,\;\; D = d_0, \tag{66}$$

$$H_1 = -A,\quad \text{and}\quad H_2 = -\frac{(B + \zeta A)}{S}. \tag{67}$$

Using the expressions in Eqs. (63) and (64) with the conditions

$$x(T) = x_\mathrm{f}\quad \text{and}\quad x'(T) = 0 \tag{68}$$

and substituting for $x(\tau)$ in Eq. (29), the requirements that $x(T) = x_\mathrm{f}$ and $x'(T) = 0$ lead to Eq. (34) where

$$p_1 = -\mathrm{e}^{-T\zeta}\alpha\cos(ST) + \mathrm{e}^{-T\zeta}\left(\frac{\alpha_1}{S} - \frac{\alpha\zeta}{S}\right)\sin(ST) - 2T\zeta + 8\zeta^2 - 2, \tag{69}$$

$$p_2 = -\mathrm{e}^{-T\zeta}\beta\cos(ST) + \mathrm{e}^{-T\zeta}\left(\frac{\beta_1}{S} - \frac{\beta\zeta}{S}\right)\sin(ST) - \gamma\zeta - 6T, \tag{70}$$

$$q_1 = \mathrm{e}^{-T\zeta}\alpha_1\cos(ST) + \mathrm{e}^{-T\zeta}\left(\left[S + \frac{\zeta^2}{S}\right]\alpha - \frac{\zeta\alpha_1}{S}\right)\sin(ST) + T - 4\zeta, \tag{71}$$

$$q_2 = \mathrm{e}^{-T\zeta}\beta_1\cos(ST) + \mathrm{e}^{-T\zeta}\left(\left[S + \frac{\zeta^2}{S}\right]\beta - \frac{\zeta\beta_1}{S}\right)\sin(ST) + \frac{\gamma}{2} + 6, \tag{72}$$

$$r_1 = -\frac{2\zeta\mathrm{e}^{-T\zeta}\cos(ST)}{T} + \frac{(1-2\zeta^2)\mathrm{e}^{-T\zeta}\sin(ST)}{ST} + \frac{2\zeta}{T}, \tag{73}$$

$$r_2 = \frac{\mathrm{e}^{-T\zeta}\cos(ST)}{T} + \mathrm{e}^{-T\zeta}\sin(ST)\left(\frac{2\zeta S}{T} - \frac{(1-2\zeta^2)\zeta}{ST}\right) - \frac{1}{T}, \tag{74}$$

$$\alpha = 2(T\zeta + 4\zeta^2 - 1),\quad \alpha_1 = T + 4\zeta, \tag{75}$$

$$\beta = 2\zeta\left(T^2 + 12 - 24\zeta^2\right),\quad \beta_1 = T^2 - 24\zeta^2 + 6, \tag{76}$$

and

$$\gamma = 2\left(2T^2 - 12 - 12T\zeta + 24\zeta^2\right). \tag{77}$$

The determinant of the matrix A_0 (see Eq. (34)) is given by

$$\mathrm{Det}(A_0) = 2T^2 - 12T\zeta + 12 + e^{-T\zeta}\left[\frac{T^2+24\zeta^2-24}{S}T\sin(ST) + (8T^2-24)\cos(ST)\right] + (2T^2+12T\zeta+12)\mathrm{e}^{-2T\zeta}. \tag{78}$$

Noting that $S = \sqrt{1-\zeta^2}$, the Taylor series expansion of this determinant, taken at $T = 0$, is

$$\mathrm{Det}(A_0)|_{T=0} = \frac{1}{720}T^8 - \frac{1}{720}\zeta T^9 + \left(\frac{1}{1260}\zeta^2 - \frac{1}{16{,}800}\right)T^{10} + O(T^{11}). \tag{79}$$

Using symbolic computation to obtain $x'(\tau)$, we find that $x'(\tau) \geq 0$ when $0 < T \leq T_N = 2\pi$. Thus no oscillations in the response $x(\tau)$ seem to occur when $T \leq T_N = 2\pi$. The same appears to be true for $T = T_N\sqrt{1-\zeta^2} := T_D$.

Appendix 2

Taking the variation of $\hat{J}_1$we obtain

$$\begin{aligned}\delta\hat{J}_1 &= \int_0^T \{\alpha z\,\delta z + \beta w\,\delta w + \lambda_1(y - \delta x') + \lambda_2(-2\zeta\,\delta y - \delta x + \delta z - \delta y') + \lambda_3(\delta w - \delta z')\}\mathrm{d}\tau \\ &= \int_0^T \{(\lambda_1' - \lambda_2)\delta x + (\lambda_2' + \lambda_1 - 2\zeta\lambda_2)\delta y + (\lambda_3' + \lambda_2 + \alpha z)\delta z + (\lambda_3 + \beta w)\delta w\}\mathrm{d}\tau \\ &\quad - \lambda_1\delta x|_0^T - \lambda_2\delta y|_0^T - \lambda_3\delta z|_0^T .\end{aligned}$$

Noting that $\delta x(0) = \delta y(0) = \delta z(0) = \delta x(T) = \delta y(T) = \delta z(T) = 0$, and noting that the variations δx, δy, and δz under the integral sign are arbitrary, we obtain

$$\lambda_1' = \lambda_2, \quad \lambda_2' = 2\zeta\lambda_2 - \lambda_1, \lambda_3' = -\lambda_2 - \alpha z, \quad \text{and} \quad w = -\lambda_3/\beta,$$

as given in Eq. (46).

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